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TEMPERED EXPONENTIAL STABILITY FOR RANDOM SEMI-DYNAMICAL SYSTEMS

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Abstract: This paper presents an approach to formulate stability for one-side random dynamical systems. The concept of exponential stability of deterministic case of discrete skew-product semiflows is generalized to the case of tempered systems. Using this tempered concept, results extending existing exponential stability conditions for noninvertible random dynamical systems are derived.

keywords: exponential stability; dynamical systems; stochastic cocycles.

MSC2010: 37H30; 37C10; 37H10.

1 Introduction

Stability analysis is one of the key issues of time-varying linear systems. Earlier results on exponential stability for infinite dimensional systems have been obtained by Przyluski and Rolewicz in [11]. More recent works on exponential stability, respectively instability of linear time-varying systems that extend this work have been reported in [9], respectively [8]. In this line, it is worth mentioning [1] for various applications. Also, in [6], [3] it is considered the exponential stability concept for linear cocycles. Beside deterministic case, we refer the reader to [2] for details with stochastic random dynamical systems.

This work presents some new results on the exponential stability of non-invertible random dynamical systems. In doing so, the main ideea is to consider systems which are defined only on the semi-axes, the so-called one-side systems. Key arguments in this direction for considering such systems, i.e. non-invertible systems are presented in [7] and [10]. We emphasize that similar results in the deterministic framework of linear time-varying systems have been established in [9].

Notations. The notations used in this paper are in general the standard ones. $\mathbb{Z}_+$ stands for positive integers. $(X, \|\cdot\|)$ be a Banach space and $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting from X into X. $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta^n : \Omega \to \Omega$ be a measurable map preserving the probability measure $\mathbb{P}$. A random variable $\varphi : \Omega \to (0, +\infty)$ is $\theta-$invariant if $\varphi(\theta\omega) = \varphi(\omega)$, for all $\omega \in \Omega$, respectively $\tilde{\Omega} \subset \Omega$ an $\theta-$invariant subset. Also, we consider the metric semi-dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta^n)$, $n \in \mathbb{Z}_+$, which is a probability space such that $\theta^n : \Omega \to \Omega$, $n \in \mathbb{Z}_+$ satisfies the measurability properties, respectively, A linear random semi-dynamical system on a Banach space X over a measurable semi-dynamical system θ^n, $n \in \mathbb{Z}_+$ is a measurable map $\phi : \mathbb{Z}_+ \times \Omega \in \mathcal{B}(X)$. For a deeper discussion regarding these notions we refer to [4], [5]. We recall only the properties which will be used in the developments from this paper:

(a) $\phi(0, \omega) = I_X$, for all $\omega \in \Omega$;

(b) $\phi(n + m, \omega) = \phi(n, \theta^m\omega)\phi(m, \omega)$, for all $n, m \in \mathbb{Z}_+$ and $\omega \in \Omega$.

2 Tempered exponential stability

Definition 1 *The random semi-dynamical system $\phi(n, \omega)$ is called tempered exponentially stable if there exists a $\theta-$invariant random variable $\alpha : \Omega \to (0, +\infty)$, and $N : \Omega \to [1, +\infty)$ such that*

$$\|\phi(n, \omega)x\| \leq N(\omega)e^{-\alpha(\omega)n}\|x\|, \tag{1}$$

for all $\omega \in \tilde{\Omega}$, and all $(n, x) \in \mathbb{Z}_+ \times X$.

We observe that previous definition can be stated as follows

Remark 1 *The random semi-dynamical system $\phi(n, \omega)$ is tempered exponentially stable if and only if there exists a $\theta-$invariant random variable $\alpha : \Omega \to (0, +\infty)$, and $N : \Omega \to [1, +\infty)$ such that*

$$\|\phi(n + m, \omega)x\| \leq N(\theta^n\omega)e^{-\alpha(\omega)n}\|\phi(m, \omega)x\|, \tag{2}$$

for all $\omega \in \tilde{\Omega}$, and all $(n, m, x) \in \mathbb{Z}_+^2 \times X$.

Theorem 1 *The semi-dynamical system $\phi(n, \omega)$ is tempered exponentially stable if and only if there exists a $\theta-$invariant random variable $\beta : \Omega \to (0, +\infty)$ and $D : \Omega \to [1, +\infty)$ such that*

$$\sum_{k=n}^{+\infty} e^{\beta(\omega)(n-k)}N(\theta^n\omega)^{-1}\|\phi(k, \omega)x\| \leq D(\omega)\|\phi(n, \omega)x\| \tag{3}$$

for all $\omega \in \tilde{\Omega}$, and all $(n, x) \in \mathbb{Z}_+ \times X$.

We now introduce the additional function $L : \mathbb{Z}_+ \times \Omega \times X \to \mathbb{R}_+$ satisfying

$$L(n, \omega, x) \leq K(\omega)\|x\|, \tag{4}$$

for $K : \Omega \to [1, +\infty)$ a random variable, where $\omega \in \tilde{\Omega}$ and $(n, x) \in \mathbb{Z}_+ \times X$.

Theorem 2 *The semi-dynamical system $\phi(n, \omega)$ is tempered exponentially stable if and only if there exists a function $L : \mathbb{Z}_+ \times \Omega \times X \to \mathbb{R}_+$ satisfying (4), a θ-invariant random variable $\eta : \Omega \to (0, +\infty)$ such that*

$$L(0, \omega, x) - L(n, \omega, x) \geq \sum_{k=0}^{n-1} e^{-\eta(\omega)k}\|\phi(k, \omega)x\| \tag{5}$$

for all $\omega \in \tilde{\Omega}$, and all $(n, x) \in \mathbb{Z}_+^* \times X$.

3 Conclusions

Tempered exponential stability in the context of random semi-dynamical systems have been introduced. Based on this, we have derived a set of necessary and sufficient conditions. By doing so we have extended various results from deterministic case of linear cocycles over semiflows to the stochastic dynamical systems in the tempered framework.

References

[1] V. C. Aitken, H. M. Schwartz, On the exponential stability of discrete-time systems with applications in observer design, IEEE Transactions on automatic control, **39**(1994), 1959–1962.

[2] L. Arnold, Random Dynamical Systems, Springer, New York, 1998.

[3] Davor Dragičević, On the exponential stability and hyperbolicity of linear cocycles, Linear and multilinear albegra, 2019, https://doi.org/10.1080/03081087.2019.1594668

[4] T.S. Doan, Lyapunov exponents for random dynamical systems, PhD Technischen Universitat Dresden, 2009.

[5] N. D. Cong, Topological dynamics of random dynamical systems, Clarendon, Oxford, 1997.

[6] C. Pötzsche, *Geometric Theory of Discrete Nonautonomous Dynamical Systems*, Lecture Notes in Mathematics, Springer 2010.

[7] L. Zhou, K. Lu, W. Zhang, Roughness of tempered exponential dichotomies for infinite-dimensional random difference equations, Journal of Differential Equations, **254**(2013), 4024–4046.

[8] I.-L. Popa, T. Ceauşu, M. Megan, Nonuniform power instability and Lyapunov sequences, Applied Mathematics and Computation, **247** (2014), 969–975

[9] I.-L. Popa, T. Ceauşu, M. Megan, On exponential stability for linear discrete-time systems in Banach spaces, Computers & Mathematics with Applications, **63** (2012), 1497–1503

[10] I.-L. Popa, Lyapunov functions for random semi-dynamical systems in terms of tempered exponential splitting, submitted for publication

[11] K.M. Przyluski, S. Rolewicz, On stability of linear time-varying infinite-dimensional discrete-time systems, Systems and Control Letters, **4** (1984), 307–315.