Extensions of Three-Valued Paraconsistent Logics

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We first prove any [conjunctive/disjunctive] 3-valued paraconsistent logic with subclassical negation (3VPLSN)'s being defined by a unique {modulo isomorphism} [conjunctive/disjunctive] 3-valued matrix and provide effective algebraic criteria of their being *subclassical*|being *maximally* paraconsistent|having no consistent non-subclassical extension implying any [conjunctive/disjunctive] 3VPLSN's being subclassical if[f] its defining 3-valued matrix's having a 2-valued submatrix|any conjunctive/both disjunctive and subclassical/refuting *Double Negation Law* 3VPLSN's being maximally paraconsistent|any conjunctive/disjunctive subclassical 3VPLSN's having no consistent non-subclassical extension. Next, any disjunctive 3VPLSN has no proper non-classical disjunctive extension, any classical extension being disjunctive and relatively axiomatized by *Resolution* rule. Further, we provide an effective algebraic criterion of a [subclassical] 3VPLSN with lattice conjunction and disjunction's having no proper [consistent non-classical] extension but that which is relatively axiomatized by *Ex Contradictione Quodlibet* rule. Finally, any disjunctive 3VPLSN with classically-valued connectives has an infinite increasing chain of finitary extensions.

Key words: logic; calculus; matrix; extension; expansion; rule; axiom

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1 INTRODUCTION

Appearance of any non-classical (in particular, many-valueed) logic inevitably raises the problems of studying both the logic itself and those related to it (in particular, its extensions) with regard to such points as their (relative) axiomatizations as well as sound and, especially, complete semantics. In this connection, the [axiomatic] maximality of various kinds of the logic under consideration — in the sense of absence of proper [axiomatic] extensions satisfying a certain property held for the given logic — becomes especially acute.

In particular, when dealing with a *paraconsistent* (viz., refuting the *Ex Contradictione Quodlibet* rule) logic, the issue of its *maximal paraconsistency* in the sense of absence of any proper paraconsistent extension becomes especially acute. Such strong version of maximal paraconsistency — as opposed to the weak *axiomatic* one (regarding merely *axiomatic* extensions) discovered in [13] for P^1 — was first observed in [7] for the *logic of paradox* LP [6] and then for HZ [3] in [9] and has been proved for arbitrary conjunctive *subclassical* (viz., having a classical extension) three-valued paraconsistent logics in the reference [Pyn 95b] of [7] as well as comprehensively studied for arbitrary four-valued expansions of a four-valued logic in [12] with providing its effective — in case of finitely many connectives — algebraic criterion properly inherited by their *four-valued* expansions. In this paper, we provide an equally effective algebraic criterion of the maximal paraconsistency of three-valued paraconsistent logics with subclassical negation [fragment] properly inherited by their *three-valued* expansions, while any such logic is *axiomatically* maximally paraconsistent. As a consequence, we prove that any conjunctive/both subclassical and disjunctive/refuting the *Double Negation Law* three-valued paraconsistent logic with subclassical negation is maximally paraconsistent. In particular, any *three-valued* expansion of $LP/HZ/P^1$ is maximally paraconsistent.

Likewise, when dealing with non-classical (in particular, many-valued) logics, their connections with the classical (two-valued) one deserves a particular emphasis. In particular, this concerns the property of a non-classical logic's being subclassical equally comprehensively studied within the framework of four-valued expansions of a four-valued logic in [12] with its equally effective algebraic criterion very similar to that found here within the context of conjunctive/disjunctive three-valued paraconsistent logics with subclassical negation. (Here, we adapt [12]'s abstract conception of *classical* logic).

To mark the framework of this study, we prove that any [conjunctive/disjunctive] 3-valued paraconsistent logic with subclassical negation is defined by a unique $\{$up to isomorphism$\}$ [conjunctive/disjunctive] 3-valued matrix.

Nevertheless, the most culminating part of the paper concerns a much more advanced issue of exploration of overall lattices of extensions of three-valued paraconsistent logics with subclassical negation going back to the works [8] and [9] as well as [11] that have advanced much the maximal paraconsistency results for LP, HZ as well as both LA [1] and its bounded expansion towards proving the fact the lattices of their extensions form four-element chains, the greatest/least consistent proper extension being relatively axiomatized by either the *Modus Ponens* rule for the *material* implication or the *Resolution* rule/the *Ex Contradictione Quodlibet* rule and being classical/defined by the direct product of any defining three-valued matrix and its two-valued submatrix. On the other hand, such does not hold for arbitrary (even both subclassical, conjunctive and disjunctive) three-valued paraconsistent logics with subclassical negation, a most representative example being P^1 [13] having infinitely many (even finitary) extensions, proved here for arbitrary disjunctive three-valued paraconsistent logics with subclassical negation and classically-valued conectives, P^1 being a term-wise definitionally minimal instance of such a kind. This inevitably raises the question: what does unify the above miscellaneous instances? In this connection, it is remarkable that, though the work [11] has unified HZ, LA and its bounded expansion, the very first instance of such a kind — the logic of paradox LP — has proved beyond the mentioned general study. Therefore, thus far, the problem raised remained still open. Here, we study it within the framework of three-valued paraconsistent logics with subclassical negation as well as chain-lattice-based conjunction and disjunction with providing an effective — in case of finitely many connectives — criterion of having the mentioned structure of extensions positively covering those subclassical logics of the kind involved which satisfy the *Contradiction Negation* axiom (in particular, the *Double Negation Law*, including arbitrary expansions of LP — such as both LA and its bounded expansion — as well as of HZ).

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra, Model Theory and Mathematical Logic not specified here explicitly, to be found, e.g., in standard mathematical handbooks like [5]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of logical and algebraic folklore. Then, in Section 3 we elaborate quite useful generic tools concerning weakly conjunctive matrices with a single non-distinguished value as well as both an enhancement of the conception of equality determinant going back to [10] and axiomatic [resp., disjunctive] extensions of logics defined by [finitely many finite disjunctive] matrices. In Sections 4, 6, 7, 8, 9 and 10 we formulate and prove main *general* results of the paper, exemplifying these by brief discussing certain representative instances of 3VPLSN.

2 BASIC ISSUES

2.1 Set-theoretical background

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations, while singletons are identified with their unique elements.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Further, given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in the sequence $\bar{t}$ form, its s-th component (viz., the value under argument s), where $s \in S$, being written as t_s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s\in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s\in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a,a\rangle | a \in S\}$, functions of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i\in(\omega\setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\operatorname{dom}\bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}\restriction(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ\langle n \times \{f\}, \Delta_D\rangle) : S \to S$. Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi_S^T \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ *of* T *in* S.

In general, we adopt the following standard notations for elements of 2^2:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle.$$

Moreover, by $\sqsubseteq$ we denote the partial ordering on 2^2 defined by $(\bar{a} \sqsubseteq \bar{b}) \stackrel{\text{def}}{\Longleftrightarrow} ((a_0 \leqslant b_0) \& (b_1 \leqslant a_1))$, for all $\bar{a}, \bar{b} \in 2^2$. Then, given any $B \subseteq 2^2$, any $f : B^n \to B$, where $n \in \omega$, is said to be *regular*, provided, for all $\bar{a}, \bar{b} \in B^n$ such that, for every $i \in n$, $a_i \sqsubseteq b_i$, it holds that $f(\bar{a}) \sqsubseteq f(\bar{b})$.

2.2 Algebraic background

Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives*. Given any $\alpha \in \wp_{\infty \setminus 1}(\omega)$, put $V_\alpha \triangleq \{x_\beta | \beta \in \alpha\}$, elements of which being viewed as *(propositional/sentential) variables of rank* α. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, referred to as the *formula* Σ*-algebra of rank* α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ (viz., Σ-terms of rank α) being called *(propositional/sentential)* Σ*-substitutions/-formulas of rank* α. (In general, any mention of α is normally omitted, whenever $\alpha = \omega$.)

A Σ-algebra $\mathfrak{A}$ with $A \subseteq 2^2$ is said to be *regular*, whenever its primary operations are so, in which case secondary ones are so as well.

2.3 Propositional logics and matrices

A *[finitary]* Σ*-rule* is any couple $\langle \Gamma, \varphi \rangle$, where $(\Gamma \cup \{\varphi\}) \in \wp_{[\omega]}(\mathrm{Fm}^\omega_\Sigma)$, normally written in the standard sequent form $\Gamma \vdash \varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. A *(substitutional)* Σ*-instance* of it is then any Σ-rule of the form $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called Σ*-axioms* and are identified with their conclusions. A[n] *[axiomatic] (finitary)* Σ*-calculus* is then any set $\mathcal{C}$ of (finitary) Σ-rules [without premises], the set of all Σ-instances of its elements being denoted by $\mathrm{SI}_\Sigma(\mathcal{C})$.

A *(propositional/sentential)* Σ*-logic* (cf., e.g., [4]) is any closure operator C over $\mathrm{Fm}^\omega_\Sigma$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$ and all $\sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$, in which case we set $\equiv^\alpha_C \triangleq \{\langle \phi, \psi \rangle \in)(\mathrm{Fm}^\alpha_\Sigma)^2 \mid C(\phi) = C(\psi)\}$, where $\alpha \in \wp_{\infty \setminus 1}(\omega)$. This is said to be *(in)consistent*, if $C(\varnothing) \neq (=) \mathrm{Fm}_\Sigma$. Then, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied in/by* C, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of* C. Next, a Σ-logic C' is said to be a *[proper] extension of* C, whenever $C \subseteq [\subsetneq] C'$, in which case C is said to be a *[proper] sublogic of*

C'. Then, a[n axiomatic] Σ-calculus $\mathcal{C}$ is said to *axiomatize C' (relatively to C)*, if C' is the least Σ-logic (being an extension of C and) satisfying every rule in $\mathcal{C}$ [(in which case it is called an *axiomatic extension of* C, while

$$C'(X) = C(X \cup \mathrm{SI}_\Sigma(\mathcal{A})). \tag{2.1}$$

for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$)]. Furthermore, we have the finitary sublogic $C_\dashv$ of C, defined by $C_\dashv(X) \triangleq (\bigcup C[\wp_\omega(X)])$, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$, called the *finitarization of* C. Then, the extension of any finitary (in particular, diagonal) Σ-logic relatively axiomatized by a finitary Σ-calculus is a sublogic of its own finitarization, in which case it is equal to this, and so is finitary. (in particular, the Σ-logic axiomatized by a finitary Σ-calculus is finitary). Further, C is said to be *[weakly] $\bar\wedge$-conjunctive*, where $\bar\wedge$ is a (possibly, secondary) binary connective of Σ, provided $C(\phi \bar\wedge \psi)[\supseteq] = C(\{\phi, \psi\})$, where $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$. Likewise, C is said to be *$\veebar$-disjunctive*, where $\veebar$ is a (possibly, secondary) binary connective of Σ, provided $C(X \cup \{\phi \veebar \psi\}) = (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, where $(X \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}^\omega_\Sigma$, in which case the following rules:

$$x_0 \vdash (x_0 \veebar x_1), \tag{2.2}$$

$$x_1 \vdash (x_0 \veebar x_1), \tag{2.3}$$

$$(x_0 \veebar x_0) \vdash x_0 \tag{2.4}$$

are satisfied in C, and so in its extensions, while any axiomatic extension of C is $\veebar$-disjunctive, in view of (2.1). Finally, C is said to be *[(axiomatically) maximally] $\sim$-paraconsistent*, where $\sim$ is a unary connective of Σ, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1 \tag{2.5}$$

[and has no proper $\sim$-paraconsistent (axiomatic) extension].

A *(logical) Σ-matrix* (cf. [4]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^\mathcal{A} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^\mathcal{A} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding Fraktur letters [with same indices, if any].) This is said to be *n-valued/[in]consistent/truth(-non)-empty/truth-$|$false-singular*, where $n \in \omega$, provided $|A| = n/D^\mathcal{A} \neq [=]A/D^\mathcal{A} = (\neq)\varnothing/|(D^\mathcal{A}|(A \setminus D^\mathcal{A}))| \in 2$, respectively. Next, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* its *Σ'-reduct* $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle \mathfrak{A}{\restriction}\Sigma', D^\mathcal{A} \rangle$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, $\mathcal{A}$ is said to be *finite[ly generated]/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ defined by $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] \supseteq X | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$, for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.6}$$

because $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}) = \{h{\restriction}\mathrm{Fm}^{\alpha}_{\Sigma} | h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\}$, for any Σ-algebra $\mathfrak{A}$, as $A \neq \varnothing$. Then, $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic, called the *logic of* M, a Σ-logic C being said to be *[finitely-]defined by* M, provided $C(X) = \mathrm{Cn}_{\mathsf{M}}(X)$, for all $X \in \wp_{[\omega]}(\mathrm{Fm}_{\Sigma})$. A Σ-logic is said to be n*-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix, in which case it is finitary (cf. [4]).

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [5]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any Σ-rule $\Gamma \vdash \phi$ being viewed as (the universal closure of, depending upon the context) the infinitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$.

A Σ-matrix $\mathcal{A}$ is said to be a *model of* a Σ-logic C, provided C is a sublogic of the logic of $\mathcal{A}$, the class of all them being denoted by $\mathrm{Mod}(C)$. Next, $\mathcal{A}$ is said to be $\sim$*-paraconsistent*, where $\sim$ is a unary connective of Σ, whenever the logic of $\mathcal{A}$ is so. Further, $\mathcal{A}$ is said to be *[weakly]* $\diamond$*-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, the logic of $\mathcal{A}$ is [weakly] $\diamond$-conjunctive. Likewise, $\mathcal{A}$ is said to be $\diamond$*-disjunctive/implicative*, whenever $((a \notin / \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, in which case the logic of $\mathcal{A}$ is $\diamond$-disjunctive, and so is the logic of any class of $\diamond$-disjunctive Σ-matrices/resp., $\mathcal{A}$ is $\veebar_{\diamond}$-disjunctive, where $(x_0 \veebar_{\diamond} x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective]* $\{$*matrix*$\}$ *homomorphism from* $\mathcal{A}$ *[on]to* $\mathcal{B}$ is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and] $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$ ([in which case $\mathcal{B}/\mathcal{A}$ is said to be a *strict surjective* $\{$*matrix*$\}$ *homomorphic image/counter-image of* $\mathcal{A}/\mathcal{B}$]), the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Recall that $(\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [((\mathrm{img}\, h) = B) \Rightarrow](\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}) \supseteq [=]\{h \circ g | g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\})$, where $\alpha \in \wp_{\infty\setminus 1}(\omega)$, and so we have:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}}(X) \subseteq [=]\, \mathrm{Cn}^{\alpha}_{\mathcal{A}}(X)), \tag{2.7}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.8}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Then, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B}{\upharpoonright}A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$.

Let $\mathcal{A}$ be a Σ-matrix. Then, $\chi^{\mathcal{A}} \triangleq \chi^{D^{\mathcal{A}}}_{A}$ is referred to as the *characteristic function of* $\mathcal{A}$. Next, given any $\theta \in \mathrm{Con}(\mathfrak{A})$ [such that $\theta \subseteq \theta^{\mathcal{A}} \triangleq (\ker \chi^{\mathcal{A}})$], we have the *quotient* $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$ *of* $\mathcal{A}$ *by* θ, in which case we get $\nu_\theta \in \hom^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}/\theta)$.

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i \in I} \mathcal{A}_i) \triangleq \langle \prod_{i \in I} \mathfrak{A}_i, \prod_{i \in I} D^{\mathcal{A}_i} \rangle$ is called the [a] *[sub]direct product of* $\overline{\mathcal{A}}$ [whenever, for each $i \in I$, $\pi_i[B] = A_i$]. As usual, when $I = 2$, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved. Likewise, if $(\mathrm{img}\, \overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$ (and $I = 2$), where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I \triangleq (\prod_{i \in I} \mathcal{A}_i)$ [resp., $\mathcal{B}$] is called the [a] *[sub]direct I-power (square) of* $\mathcal{A}$.

Given a class M of Σ-matrices, the class of all [consistent] submatrices of members of M is denoted by $\mathbf{S}_{[*]}(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of (finite) tuples constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(\omega)}(\mathsf{M})$. As it is well-known, any logic model class is closed under both $\mathbf{P}$ and $\mathbf{S}$ (cf. (2.7)).

Lemma 2.1 (Finite Subdirect Product Lemma; cf. Lemma 2.7 of [12]**).** *Let* M *be a finite class of finite* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated model of the logic of* M*. Then,* $\mathcal{A}$ *is a strict surjective homomorphic counter-image of a strict surjective homomorphic image of a member of* $\mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))$.

Theorem 2.2 (cf. Theorem 2.8 of [12]**).** *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose [both* M *and all members of it are finite and]* $\mathbf{P}^{\mathrm{SD}}_{[\omega]}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathsf{K}$ *(in particular,* $\mathbf{S}(\mathbf{P}_{[\omega]}(\mathsf{M})) \subseteq \mathsf{K}$ *{in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{[\omega]}\langle$ *in particular,* $\mathsf{K} = \mathrm{Mod}(C)\rangle\}$*). Then,* C' *is [finitely-]defined by* $\mathrm{Mod}(C') \cap \mathsf{K}$*, and so by* $\mathrm{Mod}(C')$.

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^{\alpha}_{\Sigma'} \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$ and $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma'}, \mathfrak{Fm}^{\alpha}_{\Sigma'}) = \{h{\upharpoonright}\mathrm{Fm}^{\alpha}_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma}), h[\mathrm{Fm}^{\alpha}_{\Sigma'}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty \setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, called the Σ'*-fragment of* C, in which case C is said to be a *(Σ-)expansion of* C'. In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M}{\upharpoonright}\Sigma'$.

Classical negations, matrices and logics

Let $\sim$ be a (possibly, secondary) unary connective of Σ.

A Σ-matrix $\mathcal{A}$ is said to be *[weakly] (classically) $\sim$-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow (\sim^{\mathfrak{A}} a \notin D^{\mathcal{A}})$.

Remark 2.3. Let $\diamond$ be any (possibly, secondary) binary connective of Σ and $(x_0 \tilde{\diamond} x_1) \triangleq \sim(\sim x_0 \diamond \sim x_1)$. Then, any $\sim$-negative Σ-matrix is $\diamond$-disjunctive/-conjunctive iff it is $\tilde{\diamond}$-conjunctive/-disjunctive, respectively. $\Box$

From now on, it is supposed that $\sim \in \Sigma$.

A two-valued consistent Σ-matrix $\mathcal{A}$ is said to be *$\sim$-classical*, whenever it is $\sim$-negative, in which case it is truth-non-empty, for it is consistent, and so is both false- and truth-singular but is not $\sim$-paraconsistent.

A Σ-logic is said to be *$\sim$-[sub]classical*, whenever it is [a sublogic of] the logic of a $\sim$-classical Σ-matrix. Then, $\sim$ is called a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{2.9}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd.

3 PRELIMINARY ADVANCED KEY GENERIC ISSUES

3.1 False-singular consistent weakly conjunctive matrices

Lemma 3.1. *Let $\barwedge$ be a (possibly, secondary) binary connective of Σ, $\mathcal{A}$ a false-singular weakly $\barwedge$-conjunctive Σ-matrix, $f \in (A \setminus D^{\mathcal{A}})$, I a finite set, $\overline{\mathcal{C}}$ an I-tuple constituted by consistent submatrices of $\mathcal{A}$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $(I \times \{f\}) \in B$.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in B$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in C \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{C}_j$ is a consistent submatrix of the false-singular matrix $\mathcal{A}$, we have $f \in C_j = \pi_j[B]$. Hence, there is some $b \in B$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $a \in B$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\barwedge$-conjunctive and false-singular, we have $B \ni c \triangleq (a \barwedge^{\mathfrak{B}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $B \ni (I \times \{f\})$, as required. $\Box$

3.2 Equality determinants

A *binary equality determinant for* a class M of Σ-matrices is any Σ-calculus $\varepsilon \subseteq (\wp(\mathrm{Fm}^2_\Sigma) \times \mathrm{Fm}^2_\Sigma)$ such that the infinitary universal sentence $\forall x_0 \forall x_1((\bigwedge \varepsilon) \leftrightarrow (x_0 \approx x_1))$ is true in M. Then, according to [10], a *(unitary) equality determinant for* M is any $\Upsilon \subseteq \mathrm{Fm}^1_\Sigma$ such that $\varepsilon_\Upsilon \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \Upsilon\}$ is a binary equality determinant for M.

Example 3.2 (cf. Example 1 of [10]**).** $\{x_0\}$ is a unitary equality determinant for any both false- and truth-singular (in particular, $\sim$-classical) matrix. $\square$

Lemma 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, ε a binary equality determinant for $\mathcal{A}$ and $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, h is injective.*

Proof. Then, for any $a, b \in A$ such that $h(a) = h(b)$, we have $(a = a) \Rightarrow (\mathcal{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/a]) \Rightarrow (\mathcal{B} \models (\bigwedge \varepsilon)[x_0/h(a), x_1/h(a)]) \Rightarrow (\mathcal{B} \models (\bigwedge \varepsilon)[x_0/h(a), x_1/h(b)]) \Rightarrow (\mathcal{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/b]) \Rightarrow (a = b)$. $\square$

Lemma 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, ε a binary equality determinant for $\mathcal{B}$ and $e \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Suppose e is injective. Then, ε is a binary equality determinant for $\mathcal{A}$.*

Proof. By the well-known fact that any infinitary universal sentence, being true in $\mathcal{B}$, is so in $\mathcal{A}$, being isomorphic (under e) to $(\mathcal{B} \restriction (\mathrm{img}\, e)) \in \mathbf{S}(\mathcal{B})$. $\square$

Lemma 3.5. *Let $\mathcal{A}$ be a Σ-matrix with unitary equality determinant Υ, $\mathcal{B}$ a submatrix of $\mathcal{A}$ and $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. Then, h is diagonal.*

Proof. For any $a \in B$ and any $\upsilon \in \Upsilon$, $(\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{B}}(a) \in D^{\mathcal{B}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h(a)) = h(\upsilon^{\mathfrak{B}}(a)) \in D^{\mathcal{A}})$, and so $h(a) = a$, as required. $\square$

Lemma 3.6. *Any axiomatic binary equality determinant ε for a class M of Σ-matrices is so for $\mathbf{P}(\mathsf{M})$.*

Proof. In that case, members of M are models of the infinitary universal strict Horn theory $\varepsilon[x_1/x_0] \cup \{(\bigwedge \varepsilon) \to (x_0 \approx x_1)\}$ with equality, and so are well-known to be those of $\mathbf{P}(\mathsf{M})$, as required. $\square$

3.3 Disjunctive extensions of disjunctive finitely-valued logics

Fix any (possibly, secondary) binary connective $\veebar$ of Σ. Given any $X, Y \subseteq \mathrm{Fm}^\omega_\Sigma$, put $(X \veebar Y) \triangleq \veebar[X \times Y]$.

Lemma 3.7. *Let C be a $\veebar$-disjunctive Σ-logic. Then,*

$$(\varphi \veebar C(X \cup Y)) \subseteq C(X \cup (\varphi \veebar Y)), \tag{3.1}$$

for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, all $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and all $Y \in \wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$.

Proof. By induction on $|Y| \in \omega$. The case, when $Y = \varnothing$, is by (2.3). Now, assume $Y \neq \varnothing$. Take any $\psi \in Y$, in which case $X' \triangleq (X \cup \{\psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and $Y' \triangleq (Y \setminus \{\psi\}) \in \wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$, while $|Y'| < |Y|$, whereas $(Y' \cup X') = (X \cup Y)$, and so, by induction hypothesis, we have $(\varphi \veebar C(X \cup Y)) \subseteq C(X' \cup (\varphi \veebar Y'))$. On the other hand, by (2.2), we also have $(\varphi \veebar C(X \cup Y)) \subseteq C((X \cup \{\varphi\}) \cup (\varphi \veebar Y'))$. Thus, as $Y = (Y' \cup \{\psi\})$, the $\veebar$-disjunctivity of C yields (3.1). $\Box$

Given a Σ-rule $\Gamma \vdash \phi$ and a Σ-formula ψ, put $((\Gamma \vdash \phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi) \vdash (\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

By σ_{+1} we denote the Σ-substitution extending $[x_i/x_{i+1}]_{i \in \omega}$.

Theorem 3.8. *Let M be a [finite] class of [finite $\veebar$-disjunctive] Σ-matrices, C the logic of M, while $\mathcal{A}$ an axiomatic Σ-calculus [whereas $\mathcal{C}$ a finitary Σ-calculus]. Then, the extension C' of C relatively axiomatized by $\mathcal{C}' \triangleq (\mathcal{A}[\cup(\sigma_{+1}[\mathcal{C}] \veebar x_0)])$ is defined by $\mathsf{S} \triangleq (\mathrm{Mod}(\mathcal{A}[\cup\mathcal{C}]) \cap \mathbf{S}_*(\mathsf{M}))$ [and so is $\veebar$-disjunctive].*

Proof. First, by (2.7) [and Lemma 3.7 with $X = \varnothing$ as well as the $\veebar$-disjunctivity of every $\mathcal{A} \in \mathbf{S}_*(\mathsf{M})$, and so both that and the structurality of $\mathrm{Cn}^{\omega}_{\mathcal{A}}$], we have $\mathsf{S} = (\mathrm{Mod}(\mathcal{A})[\cap\mathrm{Mod}(\mathcal{C})] \cap \mathbf{S}_*(\mathsf{M})) \subseteq (\mathrm{Mod}(\mathcal{C}') \cap \mathbf{S}_*(\mathsf{M})) \subseteq (\mathrm{Mod}(\mathcal{C}') \cap \mathrm{Mod}(C)) = \mathrm{Mod}(C')$.

Conversely, consider any [finitary] Σ-rule $\Gamma \vdash \varphi$ not satisfied in C', in which case $\varphi \notin T \triangleq C'(\Gamma) \in (\mathrm{img}\, C') \subseteq (\mathrm{img}\, \mathrm{Cn}^{\omega}_{\mathsf{M}})$, and so [by the finiteness of $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$], there is some [finite] $\alpha \in \wp_{\omega\setminus 1}(\omega)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case $\Gamma \subseteq U \triangleq (T \cap \mathrm{Fm}^{\alpha}_{\Sigma}) \not\ni \varphi$, and so, by (2.6), $U = \mathrm{Cn}^{\alpha}_{\mathsf{M}}(U) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap \mathcal{U})$, where $\mathcal{U} \triangleq \{h^{-1}[D^{\mathcal{A}}] \supseteq U \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$ [is finite, for α as well as both M and all members of it are so]. Therefore, there is some [minimal] $S \in \mathcal{U}$ not containing φ, in which case, $\Gamma \subseteq U \subseteq S$, and so $\Gamma \vdash \varphi$ is not true in $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, S \rangle$ under $[x_i/x_i]_{i \in \alpha}$. Next, we are going to show that $\mathcal{B} \in \mathrm{Mod}(\mathcal{A}[\cup\mathcal{C}])$. For consider any $(\Delta \vdash \phi) \in (\mathcal{A}[\cup\mathcal{C}])$ and any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma})$ such that $\sigma[\Delta] \subseteq S$ as well as the following exhaustive case[s]:

- $(\Delta \vdash \phi) \in \mathcal{A}$,
  in which case $\Delta = \varnothing$, and so, as $\phi \in \mathcal{A} \subseteq \mathcal{C}'$, by the structurality of C', we have $\sigma(\phi) \in (\mathrm{Fm}^{\alpha}_{\Sigma} \cap C'(\varnothing)) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap T) = U \subseteq S$.

[• $(\Delta \vdash \phi) \in \mathcal{C}$,
in which case $((\sigma_{+1}[\Delta] \vdash \sigma_{+1}(\phi)) \veebar x_0) \in \mathcal{C}'$, and so is satisfied in C'. Then, $(\mathcal{U} \setminus \{S\}) \subseteq \mathcal{U}$ is finite, for $\mathcal{U}$ is so, in which case $n \triangleq |\mathcal{U} \setminus \{S\}| \in \omega$. Take any bijection $\overline{W} : n \to (\mathcal{U} \setminus \{S\})$. Then, for each $i \in n$, $W_n \neq S$, in which case, by the minimality of $S \in \mathcal{U} \ni W_n$, we have $W_n \nsubseteq S$, and so there is some $\xi_i \in (W_n \setminus S) \neq \varnothing$. Put $\psi \triangleq (\veebar \langle \bar{\xi}, \varphi \rangle) \in \mathrm{Fm}^{\alpha}_{\Sigma}$. Let ς be the Σ-substitution extending $[x_{i+1}/\sigma(x_i); x_0/\psi]_{i \in \omega}$. Then, $((\sigma[\Delta] \veebar \psi) \vdash (\sigma(\phi) \veebar \psi)) = \varsigma((\sigma_{+1}[\Delta] \vdash \sigma_{+1}(\phi)) \veebar x_0)$ is satisfied in C', for this is structural. Moreover, in view of the $\veebar$-disjunctivity of members of M, $(\sigma[\Delta] \veebar \psi) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap \mathcal{U}) = U \subseteq T$, in which case $(\sigma(\phi) \veebar \psi) \in (\mathrm{Fm}^{\alpha}_{\Sigma} \cap T) = U \subseteq S$, and so $\sigma(\phi) \in S$, for $\psi \notin S$.]

Thus, $\mathcal{B} \in \mathrm{Mod}(\mathcal{A}[\cup \mathcal{C}])$. On the other hand, as $S \in \mathcal{U}$, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ such that $S = h^{-1}[D^{\mathcal{A}}]$, in which case $D \triangleq (\mathrm{img}\, h)$ forms a subalgebra of $\mathfrak{A}$, and so h is a surjective strict homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \restriction D)$. In this way, by (2.7), $\Gamma \vdash \varphi$ is not true in $\mathcal{D} \in \mathsf{S}$, as required [for C' is finitary, as both C and $\mathcal{C}'$ are so]. □

Lemma 3.9. *Let C be a Σ-logic and M a finite class of finite Σ-matrices. Suppose C is finitely-defined by M. Then, C is defined by M, that is, C is finitary.*

Proof. In that case, $C' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{M}} \subseteq C$, for C' is finitary. To prove the converse is to prove that $\mathsf{M} \subseteq \mathrm{Mod}(C)$. For consider any $\mathcal{A} \in \mathsf{M}$, any $\Gamma \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\varphi \in C(\Gamma)$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma] \subseteq D^{\mathcal{A}}$. Then, $\alpha \triangleq |A| \in (\wp_{\infty \setminus 1}(\omega) \cap \omega)$. Take any bijection $e : V_\alpha \to A$ to be extended to a $g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $e^{-1} \circ (h \restriction V_\omega)$ is extended to a Σ-substitution σ, in which case $\sigma(\varphi) \in C(\sigma[\Gamma])$, for C is structural, while $\sigma[\Gamma \cup \{\varphi\}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Further, as both α, M and all members of it are finite, we have the finite set $I \triangleq \{\langle f, \mathcal{B} \rangle \mid \mathcal{B} \in \mathsf{M}, f \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B})\}$, in which case, for each $i \in I$, we set $h_i \triangleq \pi_0(i)$, $\mathcal{B}_i \triangleq \pi_1(i)$ and $\theta_i \triangleq \theta^{\mathcal{B}_i}$. Then, by (2.6), we have $\theta \triangleq \equiv^{\alpha}_{C} = \equiv^{\alpha}_{C'} = ((\mathrm{Fm}^{\alpha}_{\Sigma} \times \mathrm{Fm}^{\alpha}_{\Sigma}) \cap \bigcap_{i \in I} h_i^{-1}[\theta_i])$, in which case, for every $i \in I$, $\theta \subseteq h_i^{-1}[\theta_i] = \ker(\nu_{\theta_i} \circ h_i)$, and so $g_i \triangleq (\nu_{\theta_i} \circ h_i \circ \nu_\theta^{-1}) : (\mathrm{Fm}^{\alpha}_{\Sigma} / \theta) \to B_i$. In this way, $e \triangleq (\prod_{i \in I} g_i) : (\mathrm{Fm}^{\alpha}_{\Sigma} / \theta) \to (\prod_{i \in I} B_i)$ is injective, for $(\ker e) = ((\mathrm{Fm}^{\alpha}_{\Sigma} / \theta)^2 \cap \bigcap_{i \in I} (\ker g_i))$ is diagonal. Hence, $\mathrm{Fm}^{\alpha}_{\Sigma} / \theta$ is finite, for $\prod_{i \in I} B_i$ is so, and so is $(\sigma[\Gamma]/\theta) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma} / \theta)$. For each $c \in (\sigma[\Gamma]/\theta)$, choose any $\phi_c \in (\sigma[\Gamma] \cap \nu_\theta^{-1}[\{c\}]) \neq \varnothing$. Put $\Delta \triangleq \{\phi_c \mid c \in (\sigma[\Gamma]/\theta)\} \in \wp_\omega(\sigma[\Gamma])$. Consider any $\psi \in \sigma[\Gamma]$. Then, $\Delta \ni \phi_{\nu_\theta(\psi)} \equiv^{\omega}_{C} \psi$, in which case $\psi \in C(\Delta)$, and so $\sigma[\Gamma] \subseteq C(\Delta)$. In this way, $\sigma(\varphi) \in C(\Delta) = C'(\Delta)$, for $\Delta \in \wp_\omega(\mathrm{Fm}^{\omega}_{\Sigma})$, so, by (2.6), $\sigma(\varphi) \in \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\Delta)$. Moreover, $g[\Delta] \subseteq g[\sigma[\Gamma]] =$

$h[\Gamma] \subseteq D^{\mathcal{A}}$, and so $h(\varphi) = g(\sigma(\varphi)) \in D^{\mathcal{A}}$, as required. $\Box$

Corollary 3.10. *Let* M *be a finite class of finite* $\veebar$*-disjunctive* Σ*-matrices,* C *the logic of* M *and* C' *a* $\veebar$*-disjunctive extension of* C*. Then,* C' *is defined by* $\mathsf{S} \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(C))$.

Proof. Let $\mathcal{C}$ be the finitary Σ-calculus of all finitary Σ-rules satisfied in C', C'' the finitary Σ-logic axiomatized by $\mathcal{C}$ and $\mathsf{S}' \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(C'')) = (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{C}))$. Clearly, $C'' \subseteq \mathrm{Cn}^{\omega}_{\mathsf{S}'}$. Conversely, by Theorem 3.8 with $\mathcal{A} = \varnothing$, $\mathrm{Cn}^{\omega}_{\mathsf{S}'}$ is the extension of C relatively axiomatized by $\sigma_{+1}[\mathcal{C}] \veebar x_0$. On the other hand, by the structurality and $\veebar$-disjunctivity of C' as well as Lemma 3.7 with $X = \varnothing$, $(\sigma_{+1}[\mathcal{C}] \veebar x_0) \subseteq \mathcal{C}$. Moreover, C, being a finitary sublogic of C', is a sublogic of C'', in which case $C'' \supseteq \mathrm{Cn}^{\omega}_{\mathsf{S}'}$, and so C'' is defined by S', in which case C' is finitely-defined by S', and so is defined by S', by Lemma 3.9, in which case $C' = C''$, and so $\mathsf{S} = \mathsf{S}'$, as required. $\Box$

4 SUPER-CLASSICAL MATRICES VERSUS THREE-VALUED PARACONSISTENT LOGICS WITH SUBCLASSICAL NEGATION

From now on, fix any unary $\sim \in \Sigma$.

A Σ-matrix $\mathcal{A}$ is said to be $\sim$*-super-classical*, provided $A = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, $D^{\mathcal{A}} = \{\mathsf{b}, \mathsf{t}\}$, $\sim^{\mathfrak{A}}\langle i, i\rangle = \langle 1-i, 1-i\rangle$, for each $i \in 2$, and $\sim^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}}$, in which case it is three-valued as well as both weakly $\sim$-negative and $\sim$-paraconsistent, while $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}\upharpoonright\{\sim\}$, whereas $(\mathfrak{A}\upharpoonright\{\sim\})\upharpoonright\{\mathsf{f}, \mathsf{t}\}$ is $\sim$-classical, and so $\sim$ is a subclassical negation for the logic of $\mathcal{A}$, in view of (2.7). Thus, we have argued the routine part (viz., (ii)⇒(iii)⇒(i)) of the following preliminary marking the framework of the present paper:

Theorem 4.1. *Let* C *be a* Σ*-logic. Then, the following are equivalent:*

(i) C *is three-valued and* $\sim$*-paraconsistent, while* $\sim$ *is a subclassical negation for* C*;*

(ii) C *is three-valued, while any three-valued* Σ*-matrix defining* C *is isomorphic to a* $\sim$*-super-classical one;*

(iii) C *is defined by a* $\sim$*-super-classical* Σ*-matrix.*

Proof. Assume (i) holds. Let $\mathcal{B}$ be any three-valued Σ-matrix defining C. Define an $e : \{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \to B$ as follows. In that case, $\mathcal{B}$ is $\sim$-paraconsistent, so there are some $e(\mathsf{b}) \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}}e(\mathsf{b}) \in D^{\mathcal{B}}$ and some $e(\mathsf{f}) \in (B \setminus D^{\mathcal{B}})$, in which case $e(\mathsf{f}) \neq e(\mathsf{b})$. Next, by (2.9) with $m = 1$ and $n = 0$, there is some $e(\mathsf{t}) \in D^{\mathcal{B}}$ such that $\sim^{\mathfrak{B}} e(\mathsf{t}) \notin D^{\mathcal{B}}$, in which case $e(\mathsf{f}) \neq e(\mathsf{t}) \neq e(\mathsf{b})$. In this way, $e : \{\mathsf{f}, \mathsf{b}, \mathsf{t}\} \to B$ is injective, and so bijective, for $|B| = 3$. Hence, it is an isomorphism from $\mathcal{A} \triangleq \langle e^{-1}[\mathfrak{B}], \{\mathsf{b}, \mathsf{t}\}\rangle$ onto $\mathcal{B}$. Therefore, by (2.7), C is defined by $\mathcal{A}$. Furthermore, $\sim^{\mathfrak{A}} \mathsf{b} \in D^{\mathcal{A}}$, while $\sim^{\mathfrak{A}} \mathsf{t} \notin D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{A}} \mathsf{t} = \mathsf{f}$, and so, for proving that $\mathcal{A}$ is $\sim$-super-classical, in which case (ii) holds, it only remains to show that $\sim^{\mathfrak{A}} \mathsf{f} = \mathsf{t}$. We do it by contradiction. For suppose $\sim^{\mathfrak{A}} \mathsf{f} \neq \mathsf{t}$, in which case, as $A = \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, we have the following two exhaustive cases:

1. $\sim^{\mathfrak{A}} \mathsf{f} = \mathsf{f}$.
   This contradicts to (2.9) with $m = 0$ and $n = 1$.

2. $\sim^{\mathfrak{A}} \mathsf{f} = \mathsf{b}$.
   Then, as $\sim^{\mathfrak{A}} \mathsf{b} \in D^{\mathcal{A}} = \{\mathsf{b}, \mathsf{t}\}$, we have the following two exhaustive subcases:

   (a) $\sim^{\mathfrak{A}} \mathsf{b} = \mathsf{b}$.
   Then, $\sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} a = \mathsf{b} \in D^{\mathcal{A}}$, for each $a \in D^{\mathcal{A}} = \{\mathsf{b}, \mathsf{t}\}$. This contradicts to (2.9) with $m = 3$ and $n = 0$.

   (b) $\sim^{\mathfrak{A}} \mathsf{b} = \mathsf{t}$.
   Then, $\sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \mathsf{f} = \mathsf{f}$. This contradicts to (2.9) with $m = 0$ and $n = 3$.

Thus, anyway, we come to a contradiction, as required. □

Remark 4.2 *(*cf. Example 2 of [10]). $\{x_0, \sim x_0\}$ is a unitary equality determinant for any $\sim$-superclassical Σ-matrix. □

Throghout the rest of the paper, fix any $\sim$-super-classical Σ-matrix $\mathcal{A}$. Let C be the logic of $\mathcal{A}$ and C^{NP} the least non-$\sim$-paraconsistent extension of C (viz., that which is relatively axiomatized by (2.5)).

Lemma 4.3. *Let $\mathcal{B}$ be a $\sim$-super-classical Σ-matrix and $e \in \mathrm{hom_S}(\mathcal{A}, \mathcal{B})$. Then, e is diagonal. In particular, $\mathcal{A} = \mathcal{B}$.*

Proof. Then, $\mathcal{C} \triangleq (\mathcal{A} \restriction \{\sim\}) = (\mathcal{B} \restriction \{\sim\})$ is $\sim$-superclassical, while $e \in \mathrm{hom_S}(\mathcal{C}, \mathcal{C})$, and so Lemma 3.5 and Remark 4.2 complete the proof. □

Theorem 4.4. *Let $\mathcal{B}$ be a $\sim$-super-classical Σ-matrix. Suppose $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{B} = \mathcal{A}$.*

Proof. In that case, $\mathcal{B}$ is a finite (and so finitely-generated) $\sim$-paraconsistent model of C. Then, by Lemmas 2.1, 3.3 and Remark 4.2, there are some set

I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case $\mathcal{D}$ is both weakly $\sim$-negative and, by (2.7), is $\sim$-paraconsistent, for $\mathcal{B}$ is so, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$, in which case $c \triangleq \sim^{\mathfrak{D}} b \in D^{\mathcal{D}} \subseteq \{\mathsf{b}, \mathsf{t}\}^I$, for $\mathcal{D}$ is weakly $\sim$-negative. Then, $D \ni a = (I \times \{\mathsf{b}\})$. Consider the following complementary cases:

1. $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$,
   in which case $\sim^{\mathfrak{A}} \mathsf{b} = \mathsf{b}$, and so $\sim^{\mathfrak{D}} c = b \notin D^{\mathcal{B}}$. Hence, $J \triangleq \{i \in I \mid \pi_i(c) = \mathsf{t}\} \neq \varnothing$. Given any $\bar{a} \in A^2$, set $(a_0|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, $D \ni a = (\mathsf{b}|\mathsf{b})$, $D \ni c = (\mathsf{t}|\mathsf{b})$ and $D \ni b = (\mathsf{f}|\mathsf{b})$. Then, as $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, while $J \neq \varnothing$, $f \triangleq \{\langle d, (d|\mathsf{b})\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

2. $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
   Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\{\mathsf{b}, \varphi^{\mathfrak{A}}(\mathsf{b}), \sim^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\mathsf{b})\} = A$, and so $D \supseteq \{a, \varphi^{\mathfrak{D}}(a), \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(a)\} = \{I \times \{d\} \mid d \in A\}$. Therefore, as $I \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, $f \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Then, $(g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, and so Lemma 4.3 completes the argument. □

Corollary 4.5. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, C' is $\sim$-paraconsistent, while $\sim$ is a subclassical negation for C'. Hence, by Theorem 4.1, C' is defined by a $\sim$-super-classical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the $\sim$-super-classical Σ-matrix $\mathcal{A}' \restriction \Sigma$, and so $(\mathcal{A}' \restriction \Sigma) = \mathcal{A}$, by Theorem 4.4 completing the argument. □

5 CLASSICAL EXTENSIONS

A $(2[+1])$*-ary [*b*-relative] (weak classical) conjunction for* $\mathfrak{A}$ is any $\varphi \in \mathrm{Fm}^{2[+1]}_\Sigma$ such that both $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}[, \mathsf{b}]) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{t}, \mathsf{f}[, \mathsf{b}]) \in \{\mathsf{f}[, \mathsf{b}]\}$. (Clearly, any binary conjunction for $\mathfrak{A}$ is a ternary b-relative one.)

Lemma 5.1. *Let I be a set and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\mathcal{A}^I$. Suppose either $\mathcal{B}$ is $\sim$-negative or both $\mathfrak{A}$ has a binary conjunction and either $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ or $L_4 \triangleq (A^2 \setminus (\{\mathsf{f}, \mathsf{t}\}^2 \cup \{\mathsf{b}\}^2))$ forms a subalgebra of $\mathfrak{A}^2$. Then, the following hold:*

(i) *if $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, then $\mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\}$ is embeddable into $\mathcal{B}$;*

(ii) *if* $\{\mathsf{f},\mathsf{t}\}$ *does not form a subalgebra of* $\mathfrak{A}$*, then* L_4 *forms a subalgebra of* $\mathfrak{A}^2$*, while* $(\mathcal{A}^2{\restriction}L_4)$ *is embeddable into* $\mathcal{B}$*.*

Proof. We start from proving:

Claim 5.2. *Let* I *be a set and* $\mathcal{B}$ *a consistent non-*$\sim$*-paraconsistent submatrix of* $\mathcal{A}^I$*. Suppose* $a \triangleq (I \times \{\mathsf{f}\}) \in B$ *(that is,* $b \triangleq (I \times \{\mathsf{t}\}) \in B$*). Then, the following hold:*

(i) $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}$*;*

(ii) $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ *is embeddable into* $\mathcal{B}$*.*

Proof. (i) By contradiction. For suppose $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$, in which case $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (I \times \{\mathsf{b}\})$, and so $\{c, \sim^{\mathfrak{B}} c\} \subseteq D^{\mathcal{B}}$, that contradicts to the non-$\sim$-paraconsistency of $\mathcal{B}$, for this is consistent.

(ii) As $I \neq \varnothing$, for $\mathcal{B}$ is consistent, by (i), $\{\langle d, I \times \{d\}\rangle \mid d \in \{\mathsf{f},\mathsf{t}\}\}$ is an embedding of $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ into $\mathcal{B}$, as required. □

As $\mathcal{B}$ is consistent, $I \neq \varnothing$ and there is some $a \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$. Next, we prove that there is some non-empty $J \subseteq I$ such that $(\mathsf{t}|\mathsf{b}) \in B$, where, for every $\bar{a} \in A^2$, we set $(a_0|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. For consider the following complementary cases:

- $\mathcal{B}$ is $\sim$-negative.
  Then, $b \triangleq \sim^{\mathfrak{B}} a \in D^{\mathcal{B}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$, in which case $B \ni c \triangleq \sim^{\mathfrak{B}} b \notin D^{\mathcal{B}}$, and so $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\} \neq \varnothing$. In this way, $B \ni b = (\mathsf{t}|\mathsf{b})$.

- $\mathcal{B}$ is not $\sim$-negative.
  Then, $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{t},\mathsf{f}) \in \{\mathsf{f},\mathsf{t}\}$, for some $\varphi \in \mathrm{Fm}^2_\Sigma$. Let $K \triangleq \{i \in I \mid \pi_i(a) = \mathsf{t}\}$, $L \triangleq \{i \in I \mid \pi_i(a) = \mathsf{f}\} \neq \varnothing$, for $a \notin D^{\mathcal{B}}$. Given any $\bar{a} \in A^3$, we set $(a_0|a_1|a_2) \triangleq ((K \times \{a_0\}) \cup (L \times \{a_1\}) \cup ((I \setminus (K \cup L)) \times \{a_2\})) \in A^I$. In this way, $B \ni a = (\mathsf{t}|\mathsf{f}|\mathsf{b})$. Consider the following exhaustive subcases:

  - $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$.
    Then, $B \ni b \triangleq \sim^{\mathfrak{A}} a = (\mathsf{f}|\mathsf{t}|\mathsf{b})$. Let $x \triangleq \varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{b}) \in A$. Consider the following exhaustive subsubcases:

    * $x = \mathsf{b}$.
      Then, $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (\mathsf{f}|\mathsf{f}|\mathsf{b})$. Put $J \triangleq (K \cup L) \neq \varnothing$, for $K \neq \varnothing$. In this way, $(\mathsf{t}|\mathsf{b}) = \sim^{\mathfrak{B}} c \in B$.

* $x = \mathsf{f}$.
  Then, $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (\mathsf{f}|\mathsf{f}|\mathsf{f})$. Put $J \triangleq I \neq \varnothing$. In this way, $(\mathsf{t}|\mathsf{b}) = {\sim}^{\mathfrak{B}} c \in B$.
* $x = \mathsf{t}$.
  Then, $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (\mathsf{f}|\mathsf{f}|\mathsf{t})$, and so $B \ni {\sim}^{\mathfrak{B}} c = (\mathsf{t}|\mathsf{t}|\mathsf{f})$. Put $J \triangleq I \neq \varnothing$. Then, $(\mathsf{t}|\mathsf{b}) = {\sim}^{\mathfrak{B}} \varphi^{\mathfrak{B}}(c, {\sim}^{\mathfrak{B}} c) \in B$.

– ${\sim}^{\mathfrak{A}} \mathsf{b} = \mathsf{t}$.
  Then, $B \ni b \triangleq {\sim}^{\mathfrak{A}} a = (\mathsf{f}|\mathsf{t}|\mathsf{t})$, and so $B \ni {\sim}^{\mathfrak{B}} b = (\mathsf{t}|\mathsf{f}|\mathsf{f})$. Put $J \triangleq I \neq \varnothing$. Then, $(\mathsf{t}|\mathsf{b}) = {\sim}^{\mathfrak{B}} \varphi^{\mathfrak{B}}(b, {\sim}^{\mathfrak{B}} b) \in B$.

Further, we prove:

Claim 5.3. *Suppose* ${\sim}^{\mathfrak{B}} \mathsf{b} = \mathsf{t}$ *and* $(\mathsf{t}|\mathsf{b}) \in B$. *Then,* $(I \times \{\mathsf{t}\}) \in B$.

Proof. Consider the following complementary cases:

1. $\mathcal{B}$ is $\sim$-negative.
   Then, $(\mathsf{t}|\mathsf{b}) \in D^{\mathcal{B}}$, in which case $(\mathsf{t}|\mathsf{f}) = {\sim}^{\mathfrak{B}} {\sim}^{\mathfrak{B}} (\mathsf{t}|\mathsf{b}) \in D^{\mathcal{B}}$, and so $J = I$. In this way, $(I \times \{\mathsf{t}\}) = (\mathsf{t}|\mathsf{b}) \in B$.

2. $\mathcal{B}$ is not $\sim$-negative.
   Then, $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{t},\mathsf{f})$, for some $\varphi \in \mathrm{Fm}^2_{\Sigma}$. Moreover, $b \triangleq (\mathsf{f}|\mathsf{t}) = {\sim}^{\mathfrak{B}} (\mathsf{t}|\mathsf{b}) \in B$, and so $B \ni {\sim}^{\mathfrak{B}} b = (\mathsf{t}|\mathsf{f})$. In this way, $(I \times \{\mathsf{t}\}) = {\sim}^{\mathfrak{B}} \varphi^{\mathfrak{B}}(b, {\sim}^{\mathfrak{B}} b) \in B$. ☐

Finally, consider the respective complementary cases:

(i) $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Consider the following exhaustive subcases:

1. ${\sim}^{\mathfrak{A}} \mathsf{b} = \mathsf{t}$.
   Then, by Claims 5.2(ii) and 5.3, $\mathcal{A} \restriction \{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}$.
2. ${\sim}^{\mathfrak{A}} \mathsf{b} = \mathsf{b}$,
   in which case $b \triangleq (\mathsf{t}|\mathsf{b}) \in B \ni c \triangleq {\sim}^{\mathfrak{B}} b = (\mathsf{f}|\mathsf{b})$. Consider the following complementary subsubcases:

   (a) $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.
   Then, as $J \neq \varnothing$, $\{\langle e, (e|\mathsf{b})\rangle \mid e \in \{\mathsf{f},\mathsf{t}\}\}$ is an embedding of $\mathcal{A} \restriction \{\mathsf{f},\mathsf{t}\}$ into $\mathcal{B}$.

(b) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(\mathsf{b}) \in \{\mathsf{f},\mathsf{t}\}$, in which case $\psi^{\mathfrak{A}}(\mathsf{f}) \in \{\mathsf{f},\mathsf{t}\} \ni \psi^{\mathfrak{A}}(\mathsf{t})$, for $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so, as $|\{f,\mathsf{t}\}| = 2$, we have just the following exhaustive subsubsubcases:

- $\psi^{\mathfrak{A}}(\mathsf{b}) = \psi^{\mathfrak{A}}(\mathsf{f})$,
in which case, for some $x \in \{\mathsf{f},\mathsf{t}\}$, $(I \times \{x\}) = (x|x) = \psi^{\mathfrak{B}}(c) \in B$, and so $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}$, in view of Claim 5.2(ii).
- $\psi^{\mathfrak{A}}(\mathsf{b}) = \psi^{\mathfrak{A}}(\mathsf{t})$,
in which case, for some $x \in \{\mathsf{f},\mathsf{t}\}$, $(I \times \{x\}) = (x|x) = \psi^{\mathfrak{B}}(b) \in B$, and so $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}$, in view of Claim 5.2(ii).
- $\psi^{\mathfrak{A}}(\mathsf{t}) = \psi^{\mathfrak{A}}(\mathsf{f})$,
in which case, for some $x \in \{\mathsf{f},\mathsf{t}\}$, $(I \times \{x\}) = (x|x) = \psi^{\mathfrak{B}}(\psi^{\mathfrak{B}}(c)) \in B$, and so $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is embeddable into $\mathcal{B}$, in view of Claim 5.2(ii).

(ii) $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, in view of Claims 5.2(i) and 5.3. Therefore, as $J \neq \varnothing$, $b \triangleq (\mathsf{t}|\mathsf{b}) \in D^{\mathcal{B}} \not\ni c \triangleq \sim^{B} b = (\mathsf{f}|\mathsf{b})$. And what is more, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$, in which case $\phi \triangleq \varphi(x_0, {\sim}x_0) \in \mathrm{Fm}^1_\Sigma$ and $\phi^{\mathfrak{A}}(\mathsf{f}) = \mathsf{b}$, and so $\phi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, for, otherwise, we would have $B \ni \phi^{\mathfrak{B}}(c) = (\mathsf{b}|\mathsf{b})$, and so we would get $\sim^{\mathfrak{B}}(\mathsf{b}|\mathsf{b}) = (\mathsf{b}|\mathsf{b}) \in D^{\mathcal{B}}$, contrary to the non-$\sim$-paraconsistency and consistency of $\mathcal{B}$. In this way, $f \triangleq (\mathsf{b}|\mathsf{f}) \in \{\phi^{\mathfrak{B}}(c), \sim^{\mathfrak{B}}\phi^{\mathfrak{B}}(c)\} \subseteq B$, in which case $g \triangleq \sim^{\mathfrak{B}} f = (\mathsf{b}|\mathsf{t}) \in D^{\mathcal{B}}$, and so, by the non-$\sim$-paraconsistency and consistency of $\mathcal{B}$, we get $f = \sim^{\mathfrak{B}} g \notin D^{\mathcal{B}}$. Hence, $J \neq I$. Let us prove, by contradiction, that L_4 forms a subalgebra of $\mathfrak{A}^2$. For suppose L_4 does not form a subalgebra of $\mathfrak{A}^2$. Then, $\mathcal{B}$ is $\sim$-negative. Moreover, there is some $\xi \in \mathrm{Fm}^4_\Sigma$ such that $\xi^{\mathfrak{A}^2}(\langle\mathsf{b},\mathsf{f}\rangle, \langle\mathsf{b},\mathsf{t}\rangle, \langle\mathsf{f},\mathsf{b}\rangle, \langle\mathsf{t},\mathsf{b}\rangle) \in (A^2 \setminus L_4)$, in which case $B \ni b' \triangleq \xi^{\mathfrak{B}}(f,g,c,b) = (x|y)$, where $\langle x,y\rangle \in (A^2 \setminus L_4) = (\{\mathsf{f},\mathsf{t}\}^2 \cup \{\mathsf{b}\}^2)$, and so either $\sim^{\mathfrak{B}} b' = b' \in D^{\mathcal{B}}$, if $x = \mathsf{b} = y$, or, otherwise, in which case $x, y \in \{\mathsf{f},\mathsf{t}\}$, and so $x \neq y$, by Claim 5.2(i), neither b' nor $\sim^{\mathfrak{B}} b' = (y|x)$ is in $D^{\mathcal{B}}$, for $J \neq \varnothing \neq (I \setminus J)$. This contradicts to the $\sim$-negativity of $\mathcal{B}$. Thus, L_4 forms a subalgebra of $\mathfrak{A}^2$. Hence, as $J \neq \varnothing \neq (I \setminus J)$, $e' \triangleq \{\langle\langle w,z\rangle, (w|z)\rangle \mid \langle w,z\rangle \in L_4\}$ is an embedding of $\mathcal{A}^2\restriction L_4$ into $\mathcal{B}$. $\square$

Corollary 5.4. *Let I be a set, $\mathcal{B}$ a submatrix of $\mathcal{A}^I$, $\mathcal{D}$ a $\sim$-classical Σ-matrix and $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{D})$. Then, the following hold:*

(i) *if $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, then $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is isomorphic to $\mathcal{D}$;*

(ii) *if $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$, then $L_4 \triangleq (A^2 \setminus (\{\mathsf{f}, \mathsf{t}\}^2 \cup \{\mathsf{b}\}^2))$ forms a subalgebra of $\mathfrak{A}^2$, while $\theta^{\mathcal{A}^2{\upharpoonright}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\upharpoonright}L_4)$, whereas $(\mathcal{A}^2{\upharpoonright}L_4)/\theta^{\mathcal{A}^2{\upharpoonright}L_4}$ is isomorphic to $\mathcal{D}$.*

Proof. In that case, $\mathcal{B}$ is both $\sim$-negative and consistent, for $\mathcal{B}$ is so, and so is non-$\sim$-paraconsistent. Consider the respective complementary cases:

(i) $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, by Lemma 5.1(i), there is some $g \in \hom_{\mathrm{S}}(\mathcal{A}{\upharpoonright}\{f, \mathsf{t}\}, \mathcal{B})$, in which case $(h \circ g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}{\upharpoonright}\{f, \mathsf{t}\}, \mathcal{D})$, for any $\sim$-classical Σ-matrix has no proper submatrix, and so Example 3.2 and Lemma 3.3 complete the argument.

(ii) $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, by Lemma 5.1(ii), L_4 forms a subalgebra of $\mathfrak{A}^2$, while there is an embedding e of $\mathcal{E} \triangleq (\mathcal{A}^2{\upharpoonright}L_4)$ into $\mathcal{B}$, in which case $g \triangleq (h \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{D})$, for any $\sim$-classical Σ-matrix has no proper submatrix, and so $(\ker g) \in \mathrm{Con}(\mathfrak{E})$. On the other hand, $(\ker g) = \theta \triangleq \theta^{\mathcal{E}}$, for $\mathcal{D}$ is both false- and truth-singular, so, by the Homomorphism Theorem, $g \circ \nu_\theta^{-1}$ is an isomorphism from $\mathcal{E}/\theta$ onto $\mathcal{D}$, as required. $\square$

Theorem 5.5. *C is $\sim$-subclassical iff either of the following hold:*

(i) *$\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C;*

(ii) *L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\theta^{\mathcal{A}^2{\upharpoonright}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\upharpoonright}L_4)$, in which case $(\mathcal{A}^2{\upharpoonright}L_4)/\theta^{\mathcal{A}^2{\upharpoonright}L_4}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C.*

Proof. The "if" part is by (2.7) and the fact that the submatrices of $\mathcal{A}^{[2]}$ appearing in (i[i]), respectively, are $\sim$-classical.

Conversely, consider any $\sim$-classical model $\mathcal{D}$ of C, in which case it is finite, and so finitely-generated. Hence, by Lemmas 2.1, 3.3 and Example 3.2, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{B}$ of it, in which case this is a submatrix of $\mathcal{A}^I$, and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{D})$. Then, (2.7) and Corollary 5.4 complete the argument. $\square$

On the other hand, the item (i) of Theorem 5.5 does not exhaust all $\sim$-subclassical three-valued $\sim$-paraconsistent Σ-logics, as it ensues from:

Example 5.6. Let $i \in 2$, $w \triangleq \langle i, i \rangle$, $\Sigma \triangleq \{\uplus, \sim\}$ with binary $\uplus$, $\mathcal{B}$ the $\sim$-classical Σ-matrix with $B \triangleq 2$, $D^{\mathcal{B}} \triangleq \{1\}$ and $(j \uplus^{\mathfrak{B}} k) \triangleq i$, for all $j, k \in 2$, $\sim^{\mathfrak{A}} \mathsf{b} \triangleq \mathsf{b}$ and

$$(a \uplus^{\mathfrak{A}} b) \triangleq \begin{cases} w & \text{if } a = \mathsf{b}, \\ \mathsf{b} & \text{otherwise}, \end{cases}$$

for all $a, b \in A$. Then, we have:

$$\begin{aligned} (\langle \mathsf{b}, a \rangle \uplus^{\mathfrak{A}^2} \langle b, \mathsf{b} \rangle) &= \langle w, \mathsf{b} \rangle, \\ (\langle b, \mathsf{b} \rangle \uplus^{\mathfrak{A}^2} \langle \mathsf{b}, a \rangle) &= \langle \mathsf{b}, w \rangle, \\ (\langle \mathsf{b}, a \rangle \uplus^{\mathfrak{A}^2} \langle \mathsf{b}, b \rangle) &= \langle w, \mathsf{b} \rangle, \\ (\langle a, \mathsf{b} \rangle \uplus^{\mathfrak{A}^2} \langle b, \mathsf{b} \rangle) &= \langle \mathsf{b}, w \rangle, \end{aligned}$$

for all $a, b \in \{\mathsf{f}, \mathsf{t}\}$. Therefore, L_4 forms a subalgebra of $\mathfrak{A}^2$ and $h \triangleq \chi^{\mathcal{A}^2 \upharpoonright L_4} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}^2 \upharpoonright L_4, \mathcal{B})$, in which case $\theta^{\mathcal{A}^2 \upharpoonright L_4} = (\ker h) \in \mathrm{Con}(\mathfrak{A}^2 \upharpoonright L_4)$, and so C is $\sim$-subclassical, by Theorem 5.5. However, $(\mathsf{f} \uplus^{\mathfrak{A}} \mathsf{t}) = \mathsf{b}$, so $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. $\square$

Taking Theorem 5.5 into account, in case C is $\sim$-subclassical, the unique $\sim$-classical extension of C is denoted by C^{PC}.

6 MAXIMAL PARACONSISTENCY

First, as $\mathcal{A}$ has no proper $\sim$-paraconsistent submatrix, by Theorems 3.8 and 4.1, we immediately have:

Corollary 6.1. *Any $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is axiomatically maximally so.*

Lemma 6.2. *Let $\mathcal{B}$ be a finitely-generated $\sim$-paraconsistent model of C. Suppose either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}$ is embeddable into a strict surjective homomorphic image of $\mathcal{B}$.*

Proof. Then, by Lemma 2.1 with $\mathsf{M} = \{\mathcal{A}\}$, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, some strict surjective homomorphic image $\mathcal{E}$ of $\mathcal{B}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.7), $\mathcal{D}$ is $\sim$-paraconsistent, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, $D \ni a = (I \times \{\mathsf{b}\})$. Consider the following complementary cases:

1. $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$.
Then, $\mathfrak{A}$ has a ternary b-relative conjunction $\varphi \in \mathrm{Fm}_\Sigma^3$. Put $c \triangleq \varphi^{\mathfrak{D}}(b, {\sim}^{\mathfrak{D}} b, a) \in D$, $d \triangleq {\sim}^{\mathfrak{D}} c \in D$, $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$ and $K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{f}\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$. Given any $\bar{a} \in A^3$, set $(a_0|a_1|a_2) \triangleq ((J \times \{a_0\}) \cup (K \times \{a_1\}) \cup ((I \setminus (J \cup K)) \times \{a_2\})) \in A^I$. Then, $a = (\mathsf{b}|\mathsf{b}|\mathsf{b})$ and $b = (\mathsf{t}|\mathsf{f}|\mathsf{b})$. Consider the following exhaustive subcases:

   (a) $\varphi^{\mathfrak{A}}(\mathsf{t}, \mathsf{f}, \mathsf{b}) = \mathsf{f}$,
   in which case we have $c = (\mathsf{f}|\mathsf{f}|\mathsf{b})$ and $d = (\mathsf{t}|\mathsf{t}|b)$, and so, since $K \neq \varnothing$, while $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, $f \triangleq \{\langle e, (e|e|\mathsf{b})\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

   (b) $\varphi^{\mathfrak{A}}(\mathsf{t}, \mathsf{f}, \mathsf{b}) = \mathsf{b}$,
   in which case we have $c = (\mathsf{b}|\mathsf{f}|\mathsf{b})$ and $d = (\mathsf{b}|\mathsf{t}|\mathsf{b})$, and so, since $K \neq \varnothing$, while $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, $f \triangleq \{\langle e, (\mathsf{b}|e|\mathsf{b})\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

2. $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}_\Sigma^1$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\{\mathsf{b}, \varphi^{\mathfrak{A}}(\mathsf{b}), {\sim}^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\mathsf{b})\} = A$, and so $D \supseteq \{a, \varphi^{\mathfrak{D}}(a), {\sim}^{\mathfrak{D}}\varphi^{\mathfrak{D}}(a)\} = \{I \times \{e\} \mid e \in A\}$. Therefore, as $I \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, $f \triangleq \{\langle e, I \times \{e\}\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Then, $(g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{E})$ is injective, by Lemma 3.3 and Remark 4.2. $\square$

Theorem 6.3. *The following are equivalent [provided C is ${\sim}$-subclassical]:*

(i) *C has no proper ${\sim}$-paraconsistent [${\sim}$-subclassical] extension;*

(ii) *either $\mathfrak{A}$ has a ternary b-relative conjunction or $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$ (in particular, ${\sim}^{\mathfrak{A}}\mathsf{b} \neq \mathsf{b}$, that is, ${\sim}{\sim}x_0 \notin C(x_0)$);*

(iii) *$L_3 \triangleq \{\langle \mathsf{b}, \mathsf{b}\rangle, \langle \mathsf{f}, \mathsf{t}\rangle, \langle \mathsf{t}, \mathsf{f}\rangle\}$ does not form a subalgebra of $\mathfrak{A}^2$;*

(iv) *$\mathcal{A}$ has no truth-singular ${\sim}$-paraconsistent subdirect square;*

(v) *$\mathcal{A}^2$ has no truth-singular ${\sim}$-paraconsistent submatrix;*

(vi) *C has no truth-singular ${\sim}$-paraconsistent model.*

Proof. First, assume (ii) holds. Consider any ${\sim}$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, {\sim}x_0\}) \supseteq \{x_0, {\sim}x_0\}$, while, by the structurality of C', $\langle \mathfrak{Fm}_\Sigma^\omega, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated $\sim$-paraconsistent submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma \rangle$, in view of (2.7). Then, by Lemma 6.2 and (2.7), $\mathcal{A}$ is a model of C', and so $C' = C$. Thus, (i) holds.

Next, (iv)⇒(iii) is by the fact $\sim^{\mathfrak{A}}\mathsf{b} \in \{\mathsf{b},\mathsf{t}\}$, $(L_3 \cap \{\mathsf{b},\mathsf{t}\}^2) = \{\langle \mathsf{b},\mathsf{b}\rangle\} \neq L_3$ and $\pi_{0[+1]}[L_3] = A$, while (iv) is a particular case of (v), whereas (vi)⇒(v) is by (2.7).

Now, let $\mathcal{B} \in \mathrm{Mod}(C)$ be both $\sim$-paraconsistent and truth-singular, in which case the rule $x_0 \vdash \sim x_0$ is true in $\mathcal{B}$, and so is its logical consequence $\{x_0, x_1, \sim x_1\} \vdash \sim x_0$, not being true in $\mathcal{A}$ under $[x_0/\mathsf{t}, x_1/\mathsf{b}]$ [but true in any $\sim$-classical model $\mathcal{C}'$ of C, for $\mathcal{C}'$ is $\sim$-negative]. Thus, the logic of $\{\mathcal{B}[, \mathcal{C}']\}$ is a proper $\sim$-paraconsistent [$\sim$-subclassical] extension of C, so (i)⇒(vi).

Finally, assume $\mathfrak{A}$ has no ternary b-relative conjunction and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. In that case, $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by L_3. If $\langle \mathsf{f},\mathsf{f}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t},\mathsf{b}) = \mathsf{f} = \varphi^{\mathfrak{A}}(\mathsf{t},\mathsf{f},\mathsf{b})$, in which case it would be a ternary b-relative conjunction for $\mathfrak{A}$. Likewise, if either $\langle \mathsf{b},\mathsf{f}\rangle$ or $\langle \mathsf{f},\mathsf{b}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{t},\mathsf{b}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{t},\mathsf{f},\mathsf{b}) = \mathsf{b}$, in which case it would be a ternary b-relative conjunction for $\mathfrak{A}$. Therefore, as $\sim^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$ and $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, we conclude that $(\{\langle \mathsf{f},\mathsf{b}\rangle, \langle \mathsf{t},\mathsf{b}\rangle, \langle \mathsf{b},\mathsf{t}\rangle, \langle \mathsf{b},\mathsf{f}\rangle, \langle \mathsf{f},\mathsf{f}\rangle, \langle \mathsf{t},\mathsf{t}\rangle\} \cap B) = \varnothing$. Thus, $B = L_3$ forms a subalgebra of $\mathfrak{A}^2$. In this way, (iii)⇒(ii) holds, as required. □

Theorem 6.3(i)⇔(ii[i]) is especially useful for [effective dis]proving the maximal $\sim$-paraconsistency of C [cf. Example 8.10].

6.1 Maximal paraconsistency versus subclassical consistent extensions

Theorem 6.4. *Suppose C is $\sim$-subclassical [in particular, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case C^{PC} is defined by $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$; cf. Theorem 5.5(i)]. Then, (iii)⇔(iv)⇒(v)⇔(vi)⇒(i)⇒(ii)⇐ [⇒](iii), where:*

(i) *C has a consistent non-$\sim$-subclassical (viz, not being a sublogic of C^{PC}; cf. Theorem 5.5) extension;*

(ii) *$\mathfrak{A}$ has no binary conjunction, in which case C has a proper $\sim$-paraconsistent $\sim$-subclassical extension (cf. Theorem 6.3);*

(iii) *$L_2 \triangleq \{\langle \mathsf{f},\mathsf{t}\rangle, \langle \mathsf{t},\mathsf{f}\rangle\}$ forms a subalgebra of $(\mathfrak{A}[{\restriction}\{\mathsf{f},\mathsf{t}\}])^2$;*

(iv) *$(\mathcal{A}[{\restriction}\{\mathsf{f},\mathsf{t}\}])^2$ has a truth-empty submatrix;*

(v) *$C^{[PC]}$ has a truth-empty model;*

(vi) $C^{[PC]}$ *has no theorem.*

Proof. First of all, note that, the non-"[]"-optional versions of the items ([iii]-iv) hold if[f] the "[]"optional ones do so.

Next, assume $\mathfrak{A}$ has a binary conjunction. Consider any consistent extension C' of C. In case C' is $\sim$-paraconsistent, by Theorem 6.3, $C' = C \subseteq C^{\mathrm{PC}}$. Now, assume C' is non-$\sim$-paraconsistent. Then, as C' is consistent, we have $x_0 \notin C'(\varnothing)$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(\varnothing)\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, \mathrm{Fm}^{1}_{\Sigma} \cap C'(\varnothing)\rangle$, in view of (2.7). Hence, by Lemma 2.1, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}_{*}(\mathcal{A})^{I}$ and some subdirect product $\mathcal{D}$ of it such that $\mathcal{B}$ is a strict surjective homomorphic counter-image of a strict surjective homomorphic image of $\mathcal{D}$, in which case $\mathcal{D}$ is a consistent model of C', in view of (2.7), and so, a non-$\sim$-paraconsistent submatrix of $\mathcal{A}^{I}$. Then, by (2.7), Lemma 5.1 and Theorem 5.5, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\mathrm{PC}}$, and so (i)$\Rightarrow$(ii) holds.

Further, [as $\sim^{\mathfrak{A}}\mathsf{t} = \mathsf{f}$] (iii)$\Rightarrow$ [$\Leftarrow$](ii) as well as (iii)$\Leftrightarrow$(iv)$\Rightarrow$(v)$\Leftrightarrow$(vi) are immediate, by (2.7) and the fact that, by the structurality of any Σ-logic C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(\varnothing)\rangle$ is a model of C'.

Finally, assume (v) holds. Let $\mathcal{B}$ be a truth-empty model of C, in which case the logic of $\mathcal{B}$ is an extension of C without theorems, and so a consistent one. Moreover, the rule $x_0 \vdash x_1$ is true in $\mathcal{B}$ but is not so in any both consistent and truth-non-empty (in particular, $\sim$-classical) Σ-matrix, so (i) holds. $\Box$

As it is demonstrated by the following immediate counterexample, the item (i) of Theorem 6.4 does not hold *unconditionally*:

Example 6.5. Let $\Sigma = \{\sim\}$, in which case $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while $B = \{\langle \mathsf{f}, \mathsf{t}\rangle, \langle \mathsf{t}, \mathsf{f}\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$, and so, by Theorems 6.4 and 5.5, C, being $\sim$-subclassical, has a consistent non-$\sim$-subclassical extension. $\Box$

7 WEAKLY CONJUNCTIVE THREE-VALUED PARACONSISTENT LOGICS WITH SUBCLASICAL NEGATION

Fix (in addition to $\sim$) any (possibly, secondary) binary connective $\bar{\wedge}$ of Σ.

Example 7.1. Suppose either $\mathcal{A}$ is weakly $\bar{\wedge}$-conjunctive or both $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}\upharpoonright\{\mathsf{f}, \mathsf{t}\}$ is weakly $\bar{\wedge}$-conjunctive. Then, $(x_0 \bar{\wedge} x_1)$ is a binary conjunction for $\mathfrak{A}$. $\Box$

By Theorems 4.1, 6.3 and Example 7.1, we immediately get the following corollary, subsuming the reference [Pyn 95b] of [7]:

Corollary 7.2. *Any three-valued $\sim$-paraconsistent weakly $\bar{\wedge}$-conjunctive Σ-logic with subclassical negation $\sim$ is maximally $\sim$-paraconsistent.*

7.1 Subclassical weakly conjunctive three-valued paraconsistent logics

Remark 7.3. If $\mathcal{A}$ is weakly $\bar{\wedge}$-conjunctive, then we have $(\mathsf{f}\,\bar{\wedge}^{\mathfrak{A}}\,\mathsf{b}) = \mathsf{f} = (\mathsf{b}\,\bar{\wedge}^{\mathfrak{A}}\,\mathsf{f})$, in which case we get $(\langle \mathsf{f},\mathsf{b}\rangle \,\bar{\wedge}^{\mathfrak{A}^2}\, \langle \mathsf{b},\mathsf{f}\rangle) = \langle \mathsf{f},\mathsf{f}\rangle \notin L_4 \supseteq \{\langle \mathsf{f},\mathsf{b}\rangle, \langle \mathsf{b},\mathsf{f}\rangle\}$, and so L_4 does not form a subalgebra of $\mathfrak{A}^2$. $\square$

By Theorem 5.5 and Remark 7.3, we immediately have:

Corollary 7.4. *[Providing C is weakly $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so)] C is $\sim$-subclassical if[f] $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, C^{PC}.*

Likewise, by Theorem 6.4 and Remark 7.1, we immediately have:

Corollary 7.5. *Let C' be a consistent extension of C. Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is weakly $\bar{\wedge}$-conjunctive (in particular, $\mathcal{A}$ [viz., C] is so). Then, $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is a model of C' (i.e., C^{PC} is an extension of C'; cf. Theorem 5.5).*

Example 6.5 shows that the condition of the weak $\bar{\wedge}$-conjunctivity cannot be omitted in the formulation of Corollary 7.5.

8 DISJUNCTIVE THREE-VALUED PARACONSISTENT LOGICS WITH SUBCLASSICAL NEGATION

Fix (in addition to $\sim$ and $\bar{\wedge}$) any (possibly, secondary) binary connective $\veebar$ of Σ.

Lemma 8.1. *Let $\mathcal{B}$ be a false-singular (in particular, $\sim$-[super-]classical) Σ-matrix and C' the logic of $\mathcal{B}$. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive;*

(ii) *$\mathcal{B}$ is $\veebar$-disjunctive;*

(iii) (2.2), (2.3) *and* (2.4) *are satisfied in C' (viz., are true in $\mathcal{B}$).*

Proof. First, (ii)⇒(i)⇒(iii) are immediate. Finally, assume (iii) holds. Consider any $a, b \in B$. In case $(a/b) \in D^{\mathcal{B}}$, by (2.2)/(2.3), we have $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Now, assume $(\{a,b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, $D^{\mathcal{B}} \not\ni a = b$. Hence, by (2.4), we get $D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$, so (ii) holds, as required. $\square$

8.1 Disjunctive extensions

By C^{MP} we denote the extension of C relatively axiomatized by the *Modus Ponens* rule for the *material* implication $\sim x_0 \veebar x_1$:

$$\{x_0, \sim x_0 \veebar x_1\} \vdash x_1. \tag{8.1}$$

Likewise, by C^{R} we denote the extension of C relatively axiomatized by the *Resolution* rule:

$$\{x_0 \veebar x_1, \sim x_0 \veebar x_1\} \vdash x_1. \tag{8.2}$$

Clearly, $C^{\mathrm{NP}} \subseteq C^{\mathrm{MP}} \subseteq C^{\mathrm{R}}$, by (2.2), whenever C is $\veebar$-disjunctive. Generally speaking, the converse inclusions need not hold, as we show below.

Remark 8.2. Given any $\veebar$-disjunctive Σ-logic, by (2.4)|(2.3), applying $[x_1/x_0, x_2/x_1, x_0/x_1]|[x_1/x_0, x_0/x_1]$ to $(\sigma_{+1}(2.5) \veebar x_0)$|(8.2), any extension of C' satisfies (8.2)$|(\sigma_{+1}(2.5) \veebar x_0)$, whenever it satisfies $(\sigma_{+1}(2.5) \veebar x_0)|$(8.2). Hence, C^{R} is the extension of C relatively axiomatized by $\sigma_{+1}(2.5) \veebar x_0$. $\Box$

Theorem 8.3. *Let C' be an extension of C. Suppose C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 8.1). Then, the following are equivalent:*

(i) *C' is $\sim$-classical;*

(ii) *C' is proper, consistent and $\veebar$-disjunctive;*

(iii) *$\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and C' is defined by $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$;*

(iv) *C is $\sim$-subclassical and $C' = C^{\mathrm{PC}}$;*

(v) *$C' = C^{\mathrm{R}}$ is consistent;*

(vi) *C' is consistent, non-$\sim$-paraconsistent and $\veebar$-disjunctive.*

In particular, C^{R} is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{R}} = C^{\mathrm{PC}}$. Moreover, C has no consistent non-$\sim$-classical (in particular, $\sim$-paraconsistent) proper $\veebar$-disjunctive [in particular, axiomatic] extension.

Proof. First, (i/ii) is a particular case of (iv/vi) respectively. Next, (i)⇒(ii) is by Lemma 8.1. Further, (iii)⇒(iv) is by Theorem 5.5.

Now, assume (ii) holds. Then, by Corollary 3.10, C' is defined by some $\mathsf{S} \subseteq \mathbf{S}_*(\mathcal{A})$, in which case $\mathcal{A} \notin \mathsf{S} \neq \varnothing$. Consider any $\mathcal{B} \in \mathsf{S}$. Then, $\mathsf{f} \in B$, for $\mathcal{B}$ is consistent, in which case $\mathsf{t} = \sim^{\mathfrak{A}}\mathsf{f} \in B$, and so, as $B \neq A$, $B = \{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while $\mathsf{S} = \{\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}\}$. Thus, (iii) holds.

Furthermore, in case (iii) holds, as $\mathcal{A}$ is $\sim$-paraconsistent, $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is the only non-$\sim$-paraconsistent member of $\mathbf{S}_*(\mathcal{A})$, and so (v) is by Theorem 3.8 and Remark 8.2.

Finally, (v)⇒(vi) is by Theorem 3.8 and Remark 8.2. □

Corollary 8.4. *Suppose C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 8.1). Then, the following are equivalent:*

(i) C^{NP} *is an axiomatic extension of* C*;*

(ii) C^{NP} *is* $\veebar$*-disjunctive;*

(iii) C^{NP} *is inconsistent;*

(iv) $C^{\mathrm{NP}} = C^{\mathrm{R}}$.

Proof. First, (iii)⇒(iv) is by the inclusion $C^{\mathrm{NP}} \subseteq C^{\mathrm{R}}$. Next, (iii)⇒(i)⇒(ii) are immediate. Further, (iv)⇒(ii) is by Theorem 3.8 and Remark 8.2. Finally, (i)⇒(ii) is proved by contradiction. For suppose C^{NP} is both $\veebar$-disjunctive and consistent. Then, by Theorem 8.3(vi)⇒(iii,v), $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B} \triangleq (\mathcal{A} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})) \in \mathrm{Mod}(C)$ (cf. (2.7)) is not $\sim$-paraconsistent, for $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is $\sim$-negative, and so $\mathcal{B} \in \mathrm{Mod}(C^{\mathrm{NP}})$, while $C^{\mathrm{NP}} = C^{\mathrm{R}}$, whereas (8.2) is not true in $\mathcal{B}$ under $[x_0/\langle\mathsf{b},\mathsf{t}\rangle, x_1/\langle\mathsf{f},\mathsf{t}\rangle]$. □

8.2 Subclassical disjunctive three-valued paraconsistent logics

First of all, by Theorems 5.5 and 8.3, we immediately have the following "disjunctive" analogue of Corollary 7.4:

Corollary 8.5. *[Providing C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 8.1)] C is $\sim$-subclassical if[f] $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, C^{PC}.*

Corollary 8.6. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative (and so is $\veebar_{\sqsupset}$-disjunctive), where $\sqsupset$ is a (possibly, secondary) binary connective of Σ, and C is $\sim$-subclassical. Then, C^{PC} is a unique proper consistent axiomatic extension of C and is relatively axiomatized by the* Ex Contradictione Quodlibet *axiom:*

$$\sim x_0 \sqsupset (x_0 \sqsupset x_1). \tag{8.3}$$

Proof. In that case, by Corollary 8.5, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B} \triangleq (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\})$ defines C^{PC}. On the other hand, $\mathcal{B}$ is the only consistent proper submatrix of $\mathcal{A}$. Moreover, it, being both $\sim$-negative and $\sqsupset$-implicative, is a model of (8.3) not being true in $\mathcal{A}$ under $[x_0/\mathsf{b}, x_1/\mathsf{f}]$, for it is $\sqsupset$-implicative. Then, Theorems 3.8 and 8.3 complete the argument. □

Next, combining Remark 2.3 with Corollaries 8.5 and 7.5, we get the following "disjunctive" analogue of the latter:

Corollary 8.7. *Suppose C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 8.1) and $\sim$-subclassical. Then, any consistent extension of C is a sublogic of C^{PC}.*

Example 6.5 shows that the condition of the $\veebar$-conjunctivity cannot be omitted in the formulation of Corollary 8.7.

On the other hand, Corollary 8.5 equally ensues from Lemma 8.1 and the following interesting result:

Theorem 8.8. *C has a [$\veebar$disjunctive] $\sim$-classical extension (viz., model [cf. Lemma 8.1]) if[f] $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\upharpoonright}\{\mathsf{f}, \mathsf{t}\}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C.*

Proof. The "if"+"in which case" part is by Theorem 5.5. [Conversely, let $\mathcal{D}$ be a $\veebar$-disjunctive $\sim$-classical model of C. We prove that $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ by contradiction. For suppose $\{\mathsf{f}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, by Theorem 5.5, L_4 forms a subalgebra of $\mathfrak{A}^2$, $\mathcal{B} \triangleq (\mathcal{A}^2{\upharpoonright}L_4)$ being $\veebar$-disjunctive, for $\mathcal{D}$ is so. Therefore, as $\langle \mathsf{b}, \mathsf{t}\rangle \in D^{\mathcal{B}}$, we have $\{\langle \mathsf{b}, \mathsf{t}\rangle \veebar^{\mathfrak{B}} \langle \mathsf{f}, \mathsf{b}\rangle, \langle \mathsf{f}, \mathsf{b}\rangle \veebar^{\mathfrak{B}} \langle \mathsf{b}, \mathsf{t}\rangle\} \subseteq D^{\mathcal{B}}$, in which case we get $\{\mathsf{b} \veebar^{\mathfrak{A}} \mathsf{f}, \mathsf{f} \veebar^{\mathfrak{A}} \mathsf{b}\} \subseteq D^{\mathcal{A}}$, and so we eventually get $(\langle \mathsf{f}, \mathsf{b}\rangle \veebar^{\mathfrak{B}} \langle \mathsf{b}, \mathsf{f}\rangle) \in D^{\mathcal{B}}$. This contradicts to the fact that $(\{\langle \mathsf{f}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{f}\rangle\} \cap D^{\mathcal{B}}) = \varnothing$, as required.] $\square$

It is remarkable that the $\veebar$-disjunctivity of C is not required in the formulation of Theorem 8.8, making it the right algebraic criterion of C's being "genuinely subclassical" in the sense of having a *genuinely* (viz., functionally-complete) classical extension.

By Theorems 4.1, 6.3, Lemma 8.1, Corollary 8.5, Example 7.1 and Remark 2.3, we eventually obtain the following one more *universal* maximality result, being essentially beyond the scopes of the reference [Pyn 95b] of [7]:

Corollary 8.9. *Any three-valued $\veebar$-disjunctive $\sim$-subclassical $\sim$-paraconsistent Σ-logic is maximally $\sim$-paraconsistent.*

The following counterexample shows that the condition of being $\sim$-subclassical in the formulation of Corollary 8.9 is essential:

Example 8.10. Let $\Sigma = \{\sim[, \uplus]\}$ [where $\uplus$ is binary], while $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ [whereas:

$$(a \uplus^{\mathfrak{A}} b) = \begin{cases} a & \text{if } a = b, \\ \mathsf{b} & \text{otherwise}, \end{cases}$$

for all $a, b \in A$, in which case (2.2), (2.3) and (2.4) are true in $\mathcal{A}$, and so, by Lemma 8.1, C is $\uplus$-disjunctive, in which case this has no proper $\uplus$-disjunctive $\sim$-paraconsistent extension; cf. Theorem 8.3]. But, L_3 forms a subalgebra of $\mathfrak{A}^2$, so, by Theorem 6.3, C is not maximally $\sim$-paraconsistent [and so is not $\sim$-subclassical, by Corollary 8.9]. $\Box$

9 THREE-VALUED PARACONSISTENT LOGICS WITH SUBCLASSICAL NEGATION AND LATTICE CONJUNCTION AND DISJUNCTION

A Σ-algebra $\mathfrak{B}$ is said to be a *[distributive]* $(\overline{\wedge}, \veebar)$*-lattice*, provided it satisfies [distributive] lattice identities for $\overline{\wedge}$ and $\veebar$, that is, $\langle B, \overline{\wedge}^{\mathfrak{B}}, \veebar^{\mathfrak{B}} \rangle$ is a [distributive] lattice (in the standard algebraic sense; cf. [5]), whose partial ordering is denoted by $\leq^{\mathfrak{B}}$.

Throughout this subsection, it is supposed that:

- $\mathfrak{A}$ is a $(\overline{\wedge}, \veebar)$-lattice, in which case $\langle A, \leq^{\mathfrak{A}} \rangle$ is a chain poset for $|A| = 3$, and so $\mathfrak{A}$ is a distributive $(\overline{\wedge}, \veebar)$-lattice;
- f is the least element of the poset involved or, equivalently, $\mathcal{A}$ is $\overline{\wedge}$-conjunctive/$\veebar$-disjunctive, that is, C is so/, in view of Lemma 8.1, and so C is maximally $\sim$-paraconsistent (cf. Corollary 7.2), while it is $\sim$-subclassical iff $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case C^{PC} is defined by $\mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\}$ (cf. Corollary 7.4).

Remark 9.1. Since $\mathcal{A}$ is $\veebar$-disjunctive, while f is the least element of the poset $\langle A, \leq^{\mathfrak{A}} \rangle$, we have $(\sim(x_0 \veebar x_1) \veebar x_1) \in C(\sim x_0 \veebar x_1)$. Therefore, any extension of C satisfies (8.2), whenever it satisfies (8.1). In particular, $C^{\mathrm{MP}} = C^{\mathrm{R}}$. $\Box$

Lemma 9.2. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent subdirect product of $\overline{\mathcal{C}}$. Then, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and $\hom(\mathcal{B}, \mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\}) \neq \varnothing$.*

Proof. Then, as $\langle A, \leq^{\mathfrak{A}} \rangle$ is a chain, we have $\mathsf{b}(\leq / \geq)^{\mathfrak{A}}\mathsf{t}$. Moreover, $\sim^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}} = \{\mathsf{b}, \mathsf{t}\}$. Therefore, $\mathsf{b}(\leq / \geq)^{\mathfrak{A}} \sim^{\mathfrak{A}} \mathsf{b}$. Let us prove, by contradiction, that there is some $i \in I$ such that $\mathsf{b} \notin C_i$. For suppose, for each $i \in I$, $\mathsf{b} \in C_i$. By induction on the cardinality on any $J \subseteq I$, let us prove that there is some $a \in (B \cap \{\mathsf{f}/\mathsf{t}, \mathsf{b}\}^I)$ including $J \times \{\mathsf{b}\}$. First, in case $J = \varnothing$, by Lemma 3.1, we have $d \triangleq (I \times \{\mathsf{f}\}) \in B$, and so $(J \times \{\mathsf{b}\}) = \varnothing \subseteq a \triangleq (d / \sim^{\mathfrak{B}} d) = (I \times \{\mathsf{f}/\mathsf{t}\}) \in (B \cap \{\mathsf{f}/\mathsf{t}, \mathsf{b}\}^I)$. Now, assume $J \neq \varnothing$, in which case there is some $j \in J \subseteq I$, and so $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$.

Hence, by induction hypothesis, there is some $a \in (B \cap \{\mathsf{f}/\mathsf{t}, \mathsf{b}\}^I)$ including $K \times \{\mathsf{b}\}$. Moreover, as $j \in I$, we have $\mathsf{b} \in C_j = \pi_j[B]$, in which case there is some $b \in B$ such that $\pi_j(b) = \mathsf{b}$, and so $c \triangleq (b(\overline{\wedge}/\underline{\vee})^{\mathfrak{B}} \sim^{\mathfrak{B}} b) \in B$, while, for every $i \in I$, $\pi_i(c) = \mathsf{b}$, if $\pi_i(b) = \mathsf{b}$, and $\pi_i(c) = (\mathsf{f}/\mathsf{t})$, otherwise, in which case $c \in \{\mathsf{f}/\mathsf{t}, \mathsf{b}\}^I$, while $\pi_j(c) = \mathsf{b}$, and so, as $J = (K \cup \{j\})$, we eventually get $(J \times \{\mathsf{b}\}) \subseteq (a \veebar^{\mathfrak{B}} c) \in (B \cap \{\mathsf{f}/\mathsf{t}, \mathsf{b}\}^I)$, as required. In particular, when $J = I$, we have $a \triangleq (I \times \{\mathsf{b}\}) \in B$, in which case we get $\{a, \sim^{\mathfrak{B}} a\} \subseteq D^{\mathcal{B}}$, and so $\mathcal{B}$, being consistent, is $\sim$-paraconsistent. This contradiction shows that there is some $i \in I$ such that $\mathsf{b} \notin C_i$, in which case $h \triangleq (\pi_i \restriction B) \in \hom(\mathcal{B}, \mathcal{C}_i)$, while C_i forms a subalgebra of $\mathfrak{A}$, whereas $\mathcal{C}_i = (\mathcal{A} \restriction C_i)$. Finally, as $\mathcal{C}_i$ is consistent, in which case $\mathsf{f} \in C_i$, and so $\mathsf{t} = \sim^{\mathfrak{A}} \mathsf{f} \in C_i$, we eventually conclude that $C_i = \{\mathsf{f}, \mathsf{t}\}$, for $\mathsf{b} \notin C_i$. ☐

Theorem 9.3. *C^{NP} is consistent iff C is $\sim$-subclassical, in which case $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ and C^{NP} is defined by $\mathcal{A} \times (\mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\})$.*

Proof. First, assume C is $\sim$-subclassical.

Then, any $\sim$-classical extension of C is a both consistent and non-$\sim$-paraconsistent extension of C, and so a consistent extension of C^{NP}, in which case this is consistent too.

Moreover, by Corollary 7.4, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case we have the Σ-matrix $\mathcal{B} \triangleq (\mathcal{A} \times (\mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\}))$. Consider any finite set I, any $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and any subdirect product $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{NP}})$ of $\overline{\mathcal{C}}$, in which case $\mathcal{D}$ is not $\sim$-paraconsistent. Put $J \triangleq \hom(\mathcal{D}, \mathcal{B})$. Consider any $a \in (D \setminus D^{\mathcal{D}})$, in which case $\mathcal{D}$ is consistent, and so, by Lemma 9.2, there is some $g \in \hom(\mathcal{D}, \mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\}) \neq \varnothing$. Moreover, there is some $i \in I$, in which case $f \triangleq (\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{A})$, such that $f(a) \notin D^{\mathcal{A}}$. Then, $h \triangleq (f \times g) \in J$ and $h(a) \notin D^{\mathcal{B}}$. In this way, $(\prod \Delta_J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}^J)$. Thus, by (2.7) and Theorem 2.2, C^{NP} is finitely-defined by the six-valued $\mathcal{B}$, and so, being finitary, for both the three-valued C and (2.5) are so, is defined by $\mathcal{B}$.

Conversely, assume C^{NP} is consistent, in which case $x_0 \notin T \triangleq C^{\mathrm{NP}}(\varnothing)$, while, by the structurality of C^{NP}, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T \rangle$ is a model of C^{NP} (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B}' \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma} \rangle$, in view of (2.7). Hence, by Lemma 2.1, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, being a strict surjective homomorphic counter-image of a strict surjective homomorphic image of $\mathcal{B}'$, in which case, by (2.7), $\mathcal{D}$ is a consistent model of C^{NP}, so it is not $\sim$-paraconsistent. Thus, by Lemma 9.2 and Corollary 7.4, C is $\sim$-subclassical, as required. ☐

Lemma 9.4. *Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\sim$-subclassical; cf. Corollary 7.4). Then, ((i)⇒(ii) and) (ii)⇒(iii)⇒(iv), where:*

(i) *$\mathfrak{A}$ is regular;*

(ii) *$K_{3(+1)} \triangleq \{\langle\mathsf{f},\mathsf{f}\rangle, \langle\mathsf{b},\mathsf{f}\rangle, (\langle\mathsf{b},\mathsf{t}\rangle,)\langle\mathsf{t},\mathsf{t}\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$;*

(iii) *$\mathrm{Cn}_{\mathcal{A}\upharpoonright\{\mathsf{f},\mathsf{t}\}}(\varnothing) = \mathrm{Cn}_{\mathcal{A}}(\varnothing)$;*

(iv) *$\mathcal{A}$ is not implicative.*

Proof. (First, assume (i) holds. Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{A}^2$ generated by K_4, in which case it is a subalgebra of $\mathfrak{A} \times (\mathfrak{A}\upharpoonright\{\mathsf{f},\mathsf{t}\})$, for $\{\mathsf{f},\mathsf{t}\} = \pi_1[K_4]$ forms a subalgebra of $\mathfrak{A}$. If $\langle\mathsf{t},\mathsf{f}\rangle$ was in D, there would be some $\varphi \in \mathrm{Fm}^4_\Sigma$ such that both $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{b},\mathsf{b},\mathsf{t}) = \mathsf{t}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f},\mathsf{t},\mathsf{t}) = \mathsf{f}$, in which case, since $a \sqsubseteq \mathsf{b}$, for every $a \in \{\mathsf{f},\mathsf{t}\}$, by the regularity of $\mathfrak{A}$, we would get $\mathsf{f} \sqsubseteq \mathsf{t}$. Therefore, as $\sim^{\mathfrak{B}}(\mathsf{f}/\mathsf{t}) = (\mathsf{t}/\mathsf{f})$, we conclude that $D = K_4$, and so (ii) holds.)

Next, assume (ii) holds, in which case $(\pi_{0[+1]}\upharpoonright K_{3(+1)}) \in \hom^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}^2\upharpoonright K_{3(+1)}, \mathcal{A}[\upharpoonright\{\mathsf{f},\mathsf{t}\}])$, and so (2.7) and (2.8) yield (iii).

Finally, (iii)⇒(iv) is by (2.1) and Corollary 8.6. □

Lemma 9.5. *Suppose $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\sim$-subclassical; cf. Corollary 7.4). Then, (i)⇔(ii)⇐(iii)⇒(iv), where:*

(i) *$\sim(x_0 \mathbin{\bar{\wedge}} \sim x_0) \notin C(\varnothing)$;*

(ii) *neither $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ (that is, $C(x_0) = C(\sim\sim x_0)$) nor $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$;*

(iii) *$L_5 \triangleq ((A \times \{\mathsf{f},\mathsf{t}\}) \setminus \{\langle\mathsf{b},\mathsf{f}\rangle\})$ forms a subalgebra of $\mathfrak{A}^2$;*

(iv) *C^{NP} has a proper non-axiomatic extension being both that of C and a proper sublogic of C^{MP}, being, in its turn, an axiomatic extension of C, and so of C^{NP}.*

Proof. First, (i)⇔(ii) is immediate.

Next, if $(\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b})/(\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t})$, then we have $(\sim^{\mathfrak{A}^2}\langle\mathsf{b},\mathsf{t}\rangle/(\langle\mathsf{b},\mathsf{t}\rangle \mathbin{\bar{\wedge}}^{\mathfrak{A}^2} \langle\mathsf{t},\mathsf{f}\rangle)) = \langle\mathsf{b},\mathsf{f}\rangle \notin L_5$, in which case $L_5 \supseteq \{\langle\mathsf{b},\mathsf{t}\rangle, \langle\mathsf{t},\mathsf{f}\rangle\}$ does not form a subalgebra of $\mathfrak{A}^2$, and so (iii)⇒(ii) holds.

Further, assume (iii) holds, in which case (ii) holds too, as it has been proved above. Then, by (2.7) and Theorem 9.3, the consistent Σ-logic C' of the consistent submatrix $\mathcal{D} \triangleq (\mathcal{A}^2\upharpoonright L_5)$ of $\mathcal{B} \triangleq (\mathcal{A}^2\upharpoonright(A \times \{\mathsf{f},\mathsf{t}\}))$, defining C^{NP}, is a consistent extension of $C^{[\mathrm{NP}]}$ and so a sublogic of $C^{\mathrm{PC}} = C^{\mathrm{MP}}$ (cf. Corollary 7.5, Theorem 8.3 and Remark 9.1). Moreover, (8.1) is not true in $\mathcal{D}$

under $[x_0/\langle \mathsf{b},\mathsf{t}\rangle, x_1/\langle \mathsf{f},\mathsf{t}\rangle]$, and so C' is a proper sublogic of C^{MP}. And what is more, since, for all $a \in D = L_5$, it holds that $(\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}) \Rightarrow (a = \langle \mathsf{f},\mathsf{f}\rangle)$, while $\mathcal{A}$ is $\veebar$-disjunctive, whereas $\mathsf{f} \notin D^{\mathcal{A}}$, we conclude that

$$\{\sim x_0, x_0 \veebar x_1\} \vdash x_1 \tag{9.1}$$

is true in $\mathcal{D}$. However, (9.1) is not true in $\mathcal{B}$ under $[x_0/\langle \mathsf{b},\mathsf{f}\rangle, x_1/\langle \mathsf{f},\mathsf{t}\rangle]$, and so C' is a proper extension of $C^{[\mathrm{NP}]}$. In addition, $(\pi_0 \restriction D) \in \hom^{\mathrm{S}}(\mathcal{D},\mathcal{A})$, in which case, by (2.8), we have $C(\varnothing) \subseteq C^{\mathrm{NP}}(\varnothing) \subseteq C'(\varnothing) \subseteq C(\varnothing)$, and so C' is not an axiomatic extension of $C^{[\mathrm{NP}]}$. Finally, by (ii), $\mathcal{A}$ is $\neg$-negative, where $\neg x_0 \triangleq \sim(x_0 \mathbin{\bar\wedge} (\sim\sim x_0 \veebar \sim x_0))$, in which case it, being $\veebar$-disjunctive, is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq (\neg x_0 \veebar x_1)$, and so Corollary 8.6 completes the argument of (iv), as required. $\Box$

Lemma 9.6. *Let C' be an extension of C. Suppose* (8.1) *is not satisfied in C' and L_5 does not form a subalgebra of $\mathfrak{A}^2$ (in particular, $\sim(x_0 \mathbin{\bar\wedge} \sim x_0) \in C(\varnothing)$, i.e., either $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ — that is, $C(x_0) = C(\sim\sim x_0)$ — or $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$; cf. Lemma 9.5(iii)⇒(ii)⇔(i)). Then, C' is a sublogic of C^{NP}.*

Proof. The case, when C^{NP} is inconsistent, is evident. Otherwise, by Theorem 9.3, C is $\sim$-subclassical, in which case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, C^{NP} being defined by the submatrix $\mathcal{B} \triangleq (\mathcal{A} \times (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))$ of $\mathcal{A}^2$, and so it suffices to prove that $\mathcal{B} \in \mathrm{Mod}(C')$. On the other hand, as C' does not satisfy (8.1), by Theorem 2.2, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (8.1), in which case there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(\sim^{\mathfrak{D}} a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \mathsf{b}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{f}\} \neq \varnothing$. Put $L \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$. Then, given any $\bar{a} \in A^5$, set $(a_0|a_1|a_2|a_3|a_4) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus (L \cup J)) \times \{a_1\}) \cup ((L \setminus J) \times \{a_2\}) \cup ((L \cap J) \times \{a_3\}) \cup (K \times \{a_4\})) \in A^I$. In this way:

$$D \ni a = (\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{b}|\mathsf{b}), \tag{9.2}$$

$$D \ni b = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{f}). \tag{9.3}$$

Moreover, by Lemma 3.1, we also have:

$$D \ni f \triangleq (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}), \tag{9.4}$$

$$D \ni \sim^{\mathfrak{D}} f = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}). \tag{9.5}$$

Consider the following exhaustive (as $\sim^{\mathfrak{A}}\mathsf{b} \in D^{\mathcal{A}} = \{\mathsf{b},\mathsf{t}\}$) cases:

1. $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$.
   Then, in case $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$, by (9.2) and (9.3), we have:

$$D \ni e \triangleq (a \mathbin{\overline{\wedge}^{\mathfrak{D}}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{f}), \tag{9.6}$$

$$D \ni \sim^{\mathfrak{D}} e = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{t}), \tag{9.7}$$

$$D \ni c \triangleq (e \mathbin{\underline{\vee}^{\mathfrak{D}}} \sim^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{t}), \tag{9.8}$$

$$D \ni \sim^{\mathfrak{D}} c = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{f}). \tag{9.9}$$

   Likewise, in case $\mathsf{b}(\leq / \geq)^{\mathfrak{A}}\mathsf{t}$, by (9.2) and (9.6)/(9.3), we have:

$$D \ni d \triangleq ((e/b) \mathbin{\underline{\vee}^{\mathfrak{D}}} \sim^{\mathfrak{D}} a) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{b}), \tag{9.10}$$

$$D \ni \sim^{\mathfrak{D}} d = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{b}). \tag{9.11}$$

   Consider the following complementary subcases:

   (a) $L \subseteq J$.
   Then, since $I \supseteq K \neq \varnothing = (L \setminus J)$, by (9.4), (9.5) and (9.10), $\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.7), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

   (b) $L \nsubseteq J$.
   Then, consider the following complementary subsubcases:

      i. there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{f}) = \mathsf{t}$,
      in which case, by (9.4) and (9.11), we have:

$$D \ni \varphi^{\mathfrak{D}}(\sim^{\mathfrak{D}} d, f) = (\mathsf{f}|\mathsf{f}|\mathsf{t}|\mathsf{f}|\mathsf{f}), \tag{9.12}$$

$$D \ni \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(\sim^{\mathfrak{D}} d, f) = (\mathsf{t}|\mathsf{t}|\mathsf{f}|\mathsf{t}|\mathsf{t}). \tag{9.13}$$

      Then, since $(L \setminus J) \neq \varnothing \neq K$, taking (9.4), (9.5), (9.10), (9.11), (9.12) and (9.13) into account, we see that

$$\{\langle\langle g, h\rangle, (g|g|h|g|g)\rangle \mid \langle g, h\rangle \in B\}$$

      is an embedding of $\mathcal{B}$ into $\mathcal{D}$, and so, by (2.7), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so.

      ii. there is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{f}) = \mathsf{t}$,
      Then, $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$, for, otherwise, we would have $\mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$, in which case we would get $\varphi^{\mathfrak{A}}(\mathsf{b}, \mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{f}) = \mathsf{t}$, where $\varphi \triangleq \sim(x_0 \mathbin{\overline{\wedge}} \sim x_1) \in \mathrm{Fm}^2_\Sigma$. Consider the following complementary subsubsubcases:

A. $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) = \varnothing$. Then, taking (9.6), (9.7), (9.8), (9.9), (9.10) and (9.11) into account, as $K \neq \varnothing \neq (L \setminus J)$, we conclude that $\{\langle\langle g,h\rangle, (\mathsf{b}|\mathsf{b}|h|\mathsf{b}|g)\rangle \mid \langle g,h\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, and so, by (2.7), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so.

B. $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing$. Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{B} \times \mathfrak{A}$ generated by $(B \dotplus 2) \triangleq ((B \times \{\mathsf{b}\}) \cup \{\langle\langle i,i\rangle, i\rangle \mid i \in \{\mathsf{f},\mathsf{t}\}\})$. Then, as $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (9.4), (9.5), (9.6), (9.7), (9.8), (9.9), (9.10) and (9.11), we see that $\{\langle\langle\langle g,h\rangle, j\rangle, (j|j|h|j|g)\rangle \mid \langle\langle g,h\rangle, j\rangle \in G\}$ is an embedding of $\mathcal{G} \triangleq ((\mathcal{B} \times \mathcal{A}){\restriction}G$ into $\mathcal{D}$, in which case, by (2.7), $\mathcal{G}$ is a model of C', for $\mathcal{D}$ is so. Let us prove, by contradiction, that $((D^{\mathcal{B}} \times \{\mathsf{f}\}) \cap G) = \varnothing$. For suppose $((D^{\mathcal{B}} \times \{\mathsf{f}\}) \cap G) \neq \varnothing$. Then, there is some $\psi \in \mathrm{Fm}^8_\Sigma$ such that $\psi^{\mathfrak{A}}(\mathsf{t},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{t},\mathsf{t},\mathsf{t},\mathsf{t},\mathsf{f},\mathsf{f},\mathsf{f},\mathsf{f}) = \mathsf{t}$, for $\pi_1[D^{\mathcal{B}}] = \{\mathsf{t}\}$. Let $\varphi \triangleq (\sim x_1, \sim x_0, \sim x_0, \sim x_0, x_0, x_0, x_0, x_1) \in \mathrm{Fm}^2_\Sigma$. Then, $\varphi^{\mathfrak{A}}(\mathsf{b},\mathsf{f}) = \mathsf{f}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f}) = \mathsf{t}$. This contradiction shows that $((D^{\mathcal{B}} \times \{\mathsf{f}\}) \cap G) = \varnothing$, in which case $(\pi_0{\restriction}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{B})$, and so, by (2.7), $\mathcal{B}$ is a model of C', for $\mathcal{G}$ is so.

2. $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$,
   Consider the following exhaustive (as $\langle A, \leq^{\mathfrak{A}}\rangle$ is a chain poset) subcases:

   (a) $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$.
   Then, by (9.2) and (9.3), we get:

$$D \ni c' \triangleq (a \veebar^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{b}), \tag{9.14}$$
$$D \ni d' \triangleq \sim^{\mathfrak{D}} c' = (\mathsf{t}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{t}), \tag{9.15}$$
$$D \ni e' \triangleq \sim^{\mathfrak{D}} d' = (\mathsf{f}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{f}), \tag{9.16}$$
$$D \ni f' \triangleq (c' \mathbin{\bar{\wedge}^{\mathfrak{D}}} d') = (\mathsf{b}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{b}). \tag{9.17}$$

   Consider the following complementary subsubcases:

   i. $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) = \varnothing$.
   Then, since $I \supseteq K \neq \varnothing$, by (9.4), (9.5) and (9.14), we see

that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.7), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

ii. $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) \neq \varnothing$.
Then, as $K \neq \varnothing$, by (9.4), (9.5), (9.14), (9.15), (9.16) and (9.17), we conclude that $\{\langle\langle g, h\rangle, (g|h|h|h|g)\rangle \mid \langle g, h\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, in which case, by (2.7), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so.

(b) $\mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$.
Then, by (9.2) and (9.3), we get:

$$D \ni c'' \triangleq (a \veebar^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{b}), \tag{9.18}$$

$$D \ni d'' \triangleq \sim^{\mathfrak{D}} c'' = (\mathsf{t}|\mathsf{t}|\mathsf{f}|\mathsf{t}|\mathsf{t}), \tag{9.19}$$

$$D \ni e'' \triangleq \sim^{\mathfrak{D}} d'' = (\mathsf{f}|\mathsf{f}|\mathsf{t}|\mathsf{f}|\mathsf{f}). \tag{9.20}$$

Consider the following complementary subsubcases:

i. $L \subseteq J$.
Then, as $K \neq \varnothing = (L \setminus J)$, taking (9.4), (9.5) and (9.18) into account, we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.7), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.

ii. $L \nsubseteq J$.
Then, as L_5 does not form a subalgebra of $\mathfrak{A}^2$, and so of its subalgebra $\mathfrak{B}$, there is some $\varphi \in \mathrm{Fm}^5_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{t}) = \mathsf{b}$ and $\varphi^{\mathfrak{A}}(\mathsf{f}, \mathsf{f}, \mathsf{t}, \mathsf{t}, \mathsf{t}) = \mathsf{f}$, in which case, by (9.4), (9.5), (9.18), (9.19) and (9.20), we get:

$$D \ni f'' \triangleq \varphi^{\mathfrak{D}}(f, d'', e'', c'', \sim^{\mathfrak{D}} f) = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{b}), \tag{9.21}$$

and so, as $K \neq \varnothing \neq (L \setminus J)$, taking (9.4), (9.5), (9.18), (9.19), (9.20) and (9.21) into account, we see that

$$\{\langle\langle g, h\rangle, (g|g|h|g|g)\rangle \mid \langle g, h\rangle \in B\}$$

is an embedding of $\mathcal{B}$ into $\mathcal{D}$, in which case, by (2.7), $\mathcal{B}$ is a model of C', for $\mathcal{D}$ is so. $\Box$

Theorem 9.7. *Suppose C is [not] non-$\sim$-subclassical. Then, extensions of C form the $(2[+2])$-element chain $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^\omega_{\mathcal{A} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})} \subsetneq] C^{\mathrm{MP|R}} = [C^{\mathrm{PC}} = \mathrm{Cn}^\omega_{\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\}} \subsetneq] \mathrm{Cn}^\omega_\varnothing$, C^{NP} [not] being axiomatic/$\veebar$-disjunctive, [iff*

L_5 does not form a subalgebra of $\mathfrak{A}^2$ (in particular, $\sim(x_0 \barwedge \sim x_0) \in C(\varnothing)$, i.e., either $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ — that is, $C(x_0) = C(\sim\sim x_0)$ — or $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$), in which case C^{PC} is $\veebar$-disjunctive, while, providing $\mathcal{A}$ is $\sqsupset$-implicative, where $\sqsupset \in \mathrm{Fm}^2_\Sigma$,/ $K_{3(+1)}$ forms a subalgebra of $\mathfrak{A}^2$ (in particular, $\mathfrak{A}$ is regular), C^{PC} is relatively axiomatized by (8.3)*/ $C^{\mathrm{PC}}(\varnothing) = C(\varnothing)$, in which case C^{PC} is an axiomatic extension of C/ both proper consistent extensions of C are not axiomatic, and so C has a unique/no proper consistent axiomatic extension].*

Proof. By Theorems 8.3, 9.3, Lemmas 9.4, 9.5, 9.6, Corollaries 7.2, 7.4, 7.5, 8.4, 8.6 and Remark 9.1. $\square$

Concluding this subsection, we briefly discuss various representative instances, assuming that $\Sigma \supseteq \Sigma^{(\supset)}_{\sim[,01]} \triangleq (\{\wedge, \vee(, \supset)[, \bot, \top]\})$, where both $\vee$ and $\wedge$ (as well as $\supset$) are binary [while both $\bot$ and $\top$ are nullary, whereas $\bot^{\mathfrak{A}} = \mathsf{f}$ and $\top^{\mathfrak{A}} = \mathsf{t}$].

First of all, taking Corollary 4.5 into account, the case, when $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$, $\barwedge = \wedge$, $\veebar = \vee$ and $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$, covers arbitrary *three-valued* expansions of the $\Sigma_\sim$-*logic of paradox* LP [6] (cf. [7] for the equivalent matrix definition of it tacitly used here), including those by constants — as regular ones — (in particular, the *bounded* $\Sigma_{\sim,01}$-expansion LP_{01} of LP) as well as arbitrary *three-valued* expansions of the $\Sigma^{\supset}_{\sim}$-*logic of antinomies* LA [1], when $(\bar{a} \supset^{\mathfrak{A}} \bar{b}) = \langle \max(1 - a_0, b_0), \max(1 - a_0, b_1)\rangle$, for all $\bar{a}, \bar{b} \in A$, in which case $\mathcal{A}$ is $\supset$-implicative (in particular, the *bounded* $\Sigma^{\supset}_{\sim,01}$-expansion LA_{01} of LA). In this way, Theorem 9.7 subsumes respective results obtained originally in [7], [8] and [11] *ad hoc*. Moreover, this case covers the axiomatic extensions of arbitrary non-maximally $\sim$-paraconsistent four-valued logics studied in [12] by the *Excluded Middle Law* axiom $x_0 \vee \sim x_0$ including $L(P/A)_{[01]}$.

Likewise, taking Corollary 4.5 into account, the case, when $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{b}$ and $\mathfrak{A}$ is a $(\wedge, \vee)$-lattice with zero b and unit t (in which case $\mathcal{A}$ is neither $\wedge$-conjunctive nor $\vee$-disjunctive, though), and so a $(\barwedge, \veebar)$-lattice, where $\barwedge = \tilde{\wedge}$ and $\veebar = \tilde{\vee}$ (cf. Remark 2.3), with zero f and unit b (it is this *non-artificial* instance that warrants regarding the case, when $\mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$), in which case $\mathcal{A}$ is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq ((\sim x_0 \wedge \sim x_1) \vee x_1)$, covers arbitrary *three-valued* expansions of the $\Sigma_\sim$-logic HZ [3]. In this way, Theorem 9.7 subsumes respective results obtained originally in [9] and [11] *ad hoc*.

And what is more, the case, when $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$, in which case $\sim^{\mathfrak{A}}$ is not regular, $\barwedge = \wedge$, $\veebar = \vee$ and $\mathsf{b} \leq^{\mathfrak{A}} \mathsf{t}$ (as well as $(a \supset^{\mathfrak{A}} b) = \min\{c \in A \mid b \leqslant \max(c, a)\}$, for all $a, b \in A$), in which case, when $\Sigma = \Sigma^{(\supset)}_{\sim,01}$, $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, while $K_{3\{+1\}}$ does $\{$not$\}$ form a subalgebra of $\mathfrak{A}^2$ — it is this case that warrants involving K_3 in addition to K_4, and so $\mathcal{A}$ is not

($\supset$-)implicative, in view of Lemma 9.4, is equally covered by Theorem 9.7. In this connection, the subcase, when $\Sigma = \Sigma^{(\supset)}_{\sim,01}$, and so C is actually dual — via both the lattice duality and the truth predicate complement — to the $\Sigma_{\sim,01}$-fragment of (resp., to) Gödel's three-valued logic [2] (itself), deserves a particular emphasis. Then, $\{\mathsf{f},\mathsf{t}\}$ forms a unique subalgebra of $\mathfrak{A}$, while $\mathfrak{A}_2 \triangleq (\mathfrak{A}\restriction\{\mathsf{f},\mathsf{t}\})$ satisfies the identity:

$$(x_0 \wedge \sim x_0) \approx \bot, \tag{9.22}$$

not being true in $\mathfrak{A}$ under $[x_0/\mathsf{b}]$. Therefore, that subprevariety P_2 of the prevariety P_3 generated by $\mathfrak{A}$, which is relatively axiomatized by the (9.22), is generated by $\mathfrak{A}_2$ — the reader is referred to [8] as for the conception of *prevariety*. Moreover, $\mathfrak{A}/\mathfrak{A}_2$ is embeddable into any/ non-one-element member of $(\mathsf{P}_3 \setminus \mathsf{P}_2)/\mathsf{P}_2$, respectively. Hence, P_2 is the only subprevariety of P_3 distinct from this and containing a non-one-element algebra. On the other hand, according to Theorem 9.7, C has two distinct proper consistent extensions. In this way, as opposed to the above instances, when $D^{\mathcal{A}} = \{a \in A \mid \mathfrak{A} \models (x_0 \approx (x_0 \vee \sim x_0)[a]\}$, the general study [8] is not applicable to the one under consideration. This highlights a particular value of Theorem 9.7 as well as of the case involved, though being, to some extent, rather artificial.

After all, the following counterexample collectively with Lemma 9.5(iii)$\Rightarrow$(iv) show that the condition of L_5's not forming a subalgebra of $\mathfrak{A}^2$ cannot be omitted in the formulations of Lemma 9.6 and Theorem 9.7:

Example 9.8. Let $\Sigma = \Sigma_{\sim}$, $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$, $\overline{\wedge} = \wedge$, $\veebar = \vee$ and $\mathsf{f} \leq^{\mathfrak{A}} \mathsf{t} \leq^{\mathfrak{A}} \mathsf{b}$, in which case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\sim$-subclassical; cf. Corollary 7.4), while L_5 forms a subalgebra of $\mathfrak{A}^2$. $\square$

10 DISJUNCTIVE THREE-VALUED PARACONSISTENT LOGICS WITH SUBCLASSICAL NEGATION AND CLASSICALLY-VALUED CONNECTIVES

An n-ary, where $n \in \omega$, operation f on A is said to be *classically-valued*, if $(\operatorname{img} f) \subseteq \{\mathsf{f},\mathsf{t}\}$.

Throughout this subsection, it is supposed that C is $\veebar$-disjunctive (that is, $\mathcal{A}$ is so; cf. Lemma 8.1) and all primary operations of $\mathfrak{A}$ are classically-valued, in which case:

- $\sim^{\mathfrak{A}}\mathsf{b} = \mathsf{t}$;

- $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so C is both $\sim$-subclassical (cf. Corollary 8.5) and maximally $\sim$-paraconsistent (cf. Corollary 8.9);

- $\mathcal{A}$ is both $\neg$-negative, $\bar{\wedge}$-conjunctive and $\sqsupset$-implicative, where:

$$\begin{array}{rcl}\neg x_0 & \triangleq & \sim(x_0 \veebar x_0),\\ (x_0 \mathbin{\bar{\wedge}} x_1) & \triangleq & \neg(\neg x_0 \veebar \neg x_1),\\ (x_0 \sqsupset x_1) & \triangleq & (\neg x_0 \veebar x_1),\end{array}$$

  and so C^{PC} is an extension of any consistent extension of C (cf. Corollary 7.5) and the only proper consistent axiomatic extension of C (cf. Corollary 8.6), while $\varepsilon^{\sqsupset}_{\sim} \triangleq \{\sim^i x_j \sqsupset \sim^i x_{1-j} \mid i,j \in 2\}$ is an axiomatic binary equality determinant for $\mathcal{A}$ (cf. Remark 4.2).

It is remarkable that $\veebar^{\mathfrak{A}} = \veebar^{\mathfrak{A}}_{\sqsupset}$, while the $\supset$-implicative $\sim$-super-classical $\{\sim,\supset\}$-matrix $\mathcal{S}$ with $\sim^{\mathfrak{S}}\mathsf{b} = \mathsf{t}$ and $\supset^{\mathfrak{S}} = \sqsupset^{\mathfrak{A}}$ defines the $\{\sim,\supset\}$-logic P^1 [13]. In this way, P^1 is a term-wise definitionally minimal instance of the case under consideration.

Theorem 10.1. *There is an increasing countable chain of finitary extensions of C, and so such finitary extension of C that is not (relatively) finitely-axiomatizable, in which case this is consistent.*

Proof. We use Theorem 2.2 with $\mathsf{K} \triangleq \mathrm{Mod}(C)$ tacitly.

Let $n \in (\omega \setminus 1)$ and C_n the finitary (for C, being three-valued, is so) extension of C relatively axiomatized by the finitary rule $R_n \triangleq ((\{\sim x_i \mid i \in n\} \cup \{\veebar\langle x_i\rangle_{i\in n}\}) \vdash x_n)$. Then, as C, being $\veebar$-disjunctive, satisfies (2.3), and so does any $\mathcal{B} \in \mathsf{K}$, when R_n is not true in $\mathcal{B}$ under any $v : V_{n+1} \to B$, for every $m \in (\omega\setminus n)$, R_m is not true in $\mathcal{B}$ under $v \cup [x_j/v(x_0); x_m/v(x_n)]_{j\in(m\setminus n)}$. So, $\langle C_n\rangle_{i\in n}$ is an increasing denumerable chain of finitary extensions of C.

Claim 10.2. *For any $n \in (\omega \setminus (1(+1)))$, there is a consistent subdirect n-power $\mathcal{A}_n \in \mathrm{Mod}(C)$ of $\mathcal{A}$ such that R_n is [not] true in $\mathcal{A}_{n+1[-1]}$ (and $D^{\mathcal{A}_n} = \{n \times \{\mathsf{t}\}\}$).*

Proof. Since all primary operations of $\mathfrak{A}$ are classically-valued, the set $A_n \triangleq (\{\mathsf{f},\mathsf{t}\}^n \cup \{\{\langle i,\mathsf{b}\rangle\} \cup ((n\setminus\{i\}) \times \{\mathsf{f}\}) \mid i \in n\}) \ni (n \times \{\mathsf{f}\})$ forms a subalgebra of $\mathfrak{A}^n$, so we have the consistent (for $n \neq 0$) subdirect n-power $\mathcal{A}_n \triangleq (\mathcal{A}^n{\upharpoonright}A_n) \in \mathrm{Mod}(C)$ (cf. (2.7)) of $\mathcal{A}$ with $D^{\mathcal{A}_n} = \{n \times \{\mathsf{t}\}\}$, whenever $n \neq 1$. Then, as $\mathcal{A}$ is $\veebar$-disjunctive, R_n is not true in $\mathcal{A}_n$ under $[x_i/(\{\langle i,\mathsf{b}\rangle\} \cup ((n\setminus\{i\}) \times \{\mathsf{f}\})); x_n/(n \times \{\mathsf{f}\})]_{i\in n}$ but is true in $\mathcal{A}_{n+1}$. $\square$

Then, by Claim 10.2, the increasing chain $\langle C_n \rangle_{n\in(\omega\setminus 1)}$ is injective, and so countable, in which case the finitary (for both C, being three-valued, and all R_n, $n \in (\omega \setminus 1)$, are so) extension C_ω of C relatively axiomatized by $\{R_n \mid n \in (\omega \setminus 1)\}$ is a proper extension of C_n, for any $n \in (\omega \setminus 1)$, and so, by the Compactness Theorem for classes of algebraic systems closed under ultra-products (cf. [5]) — in particular, finitary logic model classes, being universal Horn model classes axiomatized by calculi of all rules satisfied in finitary logics, C_ω is not (relatively) finitely axiomatizable, as required. $\Box$

As it has been demonstrated in the previous section, the condition of $\mathfrak{A}$'s primary operations' being classically-valued cannot be omitted in the formulation of Theorem 10.1. It is remarkable that $R_1 = (2.5)$, in which case $C_1 = C^{\mathrm{NP}}$, while C_ω, being a consistent extension of C, is a sublogic of C^{PC}, and so the infinite chain involved appears intermediate between C^{NP} and C^{PC}, in contrast to Theorem 9.7. And what is more, we have:

Proposition 10.3. *There is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that the identities:*

$$\varphi(x_0, x_0) \approx x_0, \tag{10.1}$$

$$\varphi(x_0, x_1) \approx \varphi(x_1, x_0) \tag{10.2}$$

are true in $\mathfrak{A}$.

Proof. By contradiction. For suppose there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that (10.1) and (10.2) are true in $\mathfrak{A}$. Then, $\varphi \in V_2$, for, otherwise, (10.1) would not be true in $\mathfrak{A}$ under $[x_0/\mathsf{b}]$, because all its primary operations are classically-valued. However, in that case, (10.2) is not true in $\mathfrak{A}$ under $[x_0/\mathsf{f}, x_1/\mathsf{t}]$. This contradiction completes the argument. $\Box$

This makes the present section essentially disjoint with Section 9. In addition, in contrast to Lemma 9.2, we have:

Lemma 10.4. $\mathcal{B} \triangleq \mathcal{A}_2 \in \mathrm{Mod}(C^{\mathrm{MP}}) \subseteq \mathrm{Mod}(C^{\mathrm{NP}})$ *(cf. Claim 10.2) is a consistent subdirect square of $\mathcal{A}$ such that* $\mathrm{hom}(\mathcal{B}, \mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}) = \varnothing$.

Proof. Then, $\mathcal{B} \triangleq \mathcal{A}_2 \in \mathrm{Mod}(C)$ is a consistent subdirect square of $\mathcal{A}$. Moreover, as $2 \notin 2$, $D^{\mathcal{B}} = \{\langle \mathsf{t},\mathsf{t}\rangle\}$, while, for every $b \in B$, it holds that $(\sim^{\mathfrak{B}}\langle \mathsf{t},\mathsf{t}\rangle \veebar^{\mathfrak{B}} b) = (\langle \mathsf{f},\mathsf{f}\rangle \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$ implies $b \in D^{\mathcal{B}}$, in view of the $\veebar$-disjunctivity of $\mathcal{A}$ and the fact that $\mathsf{f} \notin D^{\mathcal{A}}$. Hence, (8.1) is true in $\mathcal{B}$. Finally, let us prove, by contradiction, that $\mathrm{hom}(\mathcal{B}, \mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}) = \varnothing$. For suppose $\mathrm{hom}(\mathcal{B}, \mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\}) \neq \varnothing$. Take any $h \in \mathrm{hom}(\mathcal{B}, \mathcal{A}{\upharpoonright}\{\mathsf{f},\mathsf{t}\})$, in which case $h(\langle \mathsf{t},\mathsf{t}\rangle) = \mathsf{t}$, for $\langle \mathsf{t},\mathsf{t}\rangle \in D^{\mathcal{B}}$. Therefore, if, for any $a \in \{\langle \mathsf{b},\mathsf{f}\rangle, \langle \mathsf{f},\mathsf{b}\rangle\} \subseteq B$, it did hold that $h(a) = \mathsf{t}$, we would have $\mathsf{f} = {\sim}^{\mathfrak{A}}\mathsf{t} = h({\sim}^{\mathfrak{B}} a) = h(\langle \mathsf{t}, \mathsf{t}\rangle) = \mathsf{t}$. Hence, $h(\langle \mathsf{b}, \mathsf{f}\rangle) = \mathsf{f} = h(\langle \mathsf{f}, \mathsf{b}\rangle)$. Then, we get $\mathsf{f} = (\mathsf{f} \veebar^{\mathfrak{A}} \mathsf{f}) = h(\langle \mathsf{b}, \mathsf{f}\rangle \veebar^{\mathfrak{B}} \langle \mathsf{f}, \mathsf{b}\rangle) = h(\langle \mathsf{t}, \mathsf{t}\rangle) = \mathsf{t}$. This contradiction completes the argument. □

As a consequence, in contrast to Theorem 9.3/both Theorem 8.3 and Remark 9.1, we get:

Corollary 10.5. *$C^{\mathrm{NP/MP}}$ is not defined by $\mathcal{D} \triangleq ((\mathcal{A} \times (\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}))/(\mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}))$./ In particular, $C^{\mathrm{MP}} \neq C^{\mathrm{R}}$ is not $\veebar$-disjunctive.*

Proof. By contradiction. For suppose $C^{\mathrm{NP/MP}}$ is defined by $\mathcal{D}$. Then, by Lemma 10.4, $\mathcal{B} \triangleq \mathcal{A}_2 \in \mathrm{Mod}(C^{\mathrm{NP/MP}})$ is a consistent subdirect square of $\mathcal{A}$ such that $\hom(\mathcal{B}, \mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}) = \varnothing$, in which case it is finite, for A is so, and so is a finitely-generated consistent model of $C^{\mathrm{NP/MP}}$/, in which case this is consistent. Therefore, by Lemmas 2.1, 3.3, 3.4 and 3.6, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{D})^I$, some subdirect product $\mathcal{E}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{B})$, in which case $\mathcal{E}$ is consistent, for $\mathcal{B}$ is so (cf. (2.7)), and so $I \neq \varnothing$. On the other hand, by Lemmas 3.3, 3.4 and 3.6, g is injective, and so $((\pi_1/\Delta_{\{\mathsf{f},\mathsf{t}\}}) \circ \pi_i \circ g^{-1}) \in \hom(\mathcal{B}, \mathcal{A}{\restriction}\{\mathsf{f}, \mathsf{t}\}) = \varnothing$, where $i \in I \neq \varnothing$. This contradiction/ and Theorem 8.3 completes/complete the argument. □

Finally, P^1 collectively with Theorem 10.1 show that, despite of Theorem 9.7, three-valued (even both conjunctive, disjunctive and subclassical) paraconsistent logics with subclassical negation need not have finitely many (even merely finitary) extensions.

11 CONCLUSIONS

Aside from quite useful non-trivial general results and their numerous illustrative applications, the present paper (like [12]) demonstrates a special value of the conception of equality determinant initially suggested in [10] just for the sake of construction of *two-side* sequent calculi for *many*-valued logics, within the framework of *algebraic aspects of MVL*.

And what is more, the principal advance of the present study with regard to the reference [Pyn 95b] of [7] consists in proving both the maximal paraconsistency of subclassical disjunctive three-valued paraconsistent logics and inheritance of the maximal paraconsistency by *three-valued* expansions of maximally paraconsistent three-valued logics with subclassical negation, because both paraconsistency, subclassical negation and ternary b-relative conjunction are inherited by expansions, while the property of being subclassical is not, generally speaking, so.

After all, various effective algebraic criteria definitely make the paper well-related to *Soft Computing*.

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