Properties of the Robin's Inequality

Frank Vega

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ABSTRACT. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln\ln n$ where $\sigma(n)$ is the divisor function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We prove the Robin's inequality is true for every natural number $n > 5040$ when $15 \nmid n$, where $15 \nmid n$ means that n is not divisible by 15. More specifically: every counterexample should be divisible by $2^{20} \times 3^{13} \times 5^{8} \times k_1$ or either $2^{20} \times 3^{13} \times k_2$ or $2^{20} \times 5^{8} \times k_3$, where $k_1, k_2, k_3 > 1$, $2 \nmid k_1$, $3 \nmid k_1$, $5 \nmid k_1$, $2 \nmid k_2$, $3 \nmid k_2$, $2 \nmid k_3$ and $5 \nmid k_3$.

1. INTRODUCTION

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [3]. It is of great interest in number theory because it implies results about the distribution of prime numbers [3]. It was proposed by Bernhard Riemann (1859), after whom it is named [3]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [3]. The divisor function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n,

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [5]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^{\gamma} \times n \times \ln\ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [2]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [2]. Using this inequality, we show an interesting result.

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2. Results

Theorem 2.1. *Given a natural number* $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$ *such that* $p_1, p_2, \ldots, p_m$ *are prime numbers, then we obtain the following inequality*

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

Proof. For a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we obtain the following formula

$$\sigma(n) = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i - 1} \tag{2.1}$$

from the Ramanujan's notebooks [1]. In this way, we have that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)}. \tag{2.2}$$

However, for any prime power $p_i^{a_i}$, we have that

$$\frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)} < \frac{p_i^{a_i+1}}{p_i^{a_i} \times (p_i - 1)} = \frac{p_i}{p_i - 1}.$$

Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}. \tag{2.3}$$

□

Theorem 2.2. *Given some prime numbers* $p_1, p_2, \ldots, p_m$*, then we obtain the following inequality,*

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

Proof. Given a prime number p_i, we obtain that

$$\frac{p_i}{p_i - 1} = \frac{p_i^2}{p_i^2 - p_i}$$

and that would be equivalent to

$$\frac{p_i^2}{p_i^2 - p_i} = \frac{p_i^2}{p_i^2 - 1 - (p_i - 1)}$$

and that is the same as

$$\frac{p_i^2}{p_i^2 - 1 - (p_i - 1)} = \frac{p_i^2}{(p_i - 1) \times (\frac{p_i^2 - 1}{(p_i - 1)} - 1)}$$

which is equal to

$$\frac{p_i^2}{(p_i - 1) \times (\frac{p_i^2 - 1}{(p_i - 1)} - 1)} = \frac{p_i^2}{(p_i - 1) \times \frac{p_i^2 - 1}{(p_i - 1)} \times (1 - \frac{(p_i - 1)}{p_i^2 - 1})}$$

that is equivalent to

$$\frac{p_i^2}{(p_i - 1) \times \frac{p_i^2-1}{(p_i-1)} \times (1 - \frac{(p_i-1)}{p_i^2-1})} = \frac{p_i^2}{p_i^2 - 1} \times \frac{1}{1 - \frac{(p_i-1)}{p_i^2-1}}$$

which is the same as

$$\frac{p_i^2}{p_i^2 - 1} \times \frac{1}{1 - \frac{(p_i-1)}{p_i^2-1}} = \frac{1}{1 - p_i^{-2}} \times \frac{1}{1 - \frac{1}{(p_i+1)}}$$

and finally

$$\frac{1}{(1 - p_i^{-2})} \times \frac{1}{1 - \frac{1}{(p_i+1)}} = \frac{1}{(1 - p_i^{-2})} \times \frac{p_i + 1}{p_i}.$$

In this way, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} = \prod_{i=1}^{m} \frac{1}{1 - p_i^{-2}} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

However, we know that

$$\prod_{i=1}^{m} \frac{1}{1 - p_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}}$$

where p_j is the j^{th} prime number and we have that

$$\prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [5]. Consequently, we obtain that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}. \tag{2.4}$$

□

Definition 2.3. We recall that an integer n is said to be squarefree if for every prime divisor p of n we have $p^2 \nmid n$, where $p^2 \nmid n$ means that p^2 does not divide n [2].

Theorem 2.4. *Given a squarefree number $n = q_1 \times \ldots \times q_m$ such that $q_1, q_2, \ldots, q_m$ are odd prime numbers, $3 \nmid n$ and $5 \nmid n$, then we obtain the following inequality*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \ln\ln(2^{19} \times n). \tag{2.5}$$

Proof. This proof is very similar with the demonstration in Theorem 1.1 from the article reference [2]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [2]. Put $\omega(n) = m$ [2]. We need to prove the assertion for those integers with $m = 1$. From the equation (2.1), we obtain that

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \ldots \times (q_m + 1) \tag{2.6}$$

when $n = q_1 \times q_2 \times \ldots \times q_m$. In this way, for any prime number $p_i \geq 7$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{p_i}) \leq e^{\gamma} \times \ln\ln(2^{19} \times p_i). \tag{2.7}$$

For $p_i = 7$, we have that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{7}) \leq e^{\gamma} \times \ln \ln(2^{19} \times 7)$$

is actually true. For another prime number $p_i > 7$, we have that

$$(1 + \frac{1}{p_i}) < (1 + \frac{1}{7})$$

and

$$e^{\gamma} \times \ln \ln(2^{19} \times 7) < e^{\gamma} \times \ln \ln(2^{19} \times p_i)$$

which clearly implies that the inequality (2.7) is true for every prime number $p_i \geq 7$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those squarefree n with $\omega(n) = m$ [2]. So let $n = q_1 \times \ldots \times q_m$ be a squarefree number and assume that $q_1 < \ldots < q_m$ for $q_m \geq 7$.

Case 1 :$q_m \geq \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times \ln \ln(2^{19} \times q_1 q_1 \times \ldots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$
$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show that

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1}) \leq$$
$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times q_m \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \ln \ln(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} \geq$$
$$\frac{\ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})}{\ln q_m}.$$

From the reference [2], we have that if $0 < a < b$, then

$$\frac{\ln b - \ln a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{2.8}$$

We can apply the inequality (2.8) to the previous one just using $b = \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m)$ and $a = \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$. Certainly, we have that

$$\ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln(2^{19} \times q_1 \times \ldots \times q_{m-1}) =$$
$$\ln \frac{2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \ldots \times q_{m-1}} = \ln q_m.$$

In this way, we obtain that

$$\frac{q_m \times (\ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} >$$

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \ldots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \ldots \times q_m)} \geq \frac{\ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1})}{\ln q_m}$$

which is trivially true for $q_m \geq \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m)$ [2].

Case 2 :$q_m < \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \ln\ln(2^{19} \times n).$$

We know that $\frac{3}{2} < 1.6 = \frac{4 \times 6}{3 \times 5}$. Nevertheless, we could have that

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times 6 \times \sigma(n)}{3 \times 5 \times n} \times \frac{\pi^2}{6} = \frac{\sigma(3 \times 5 \times n)}{3 \times 5 \times n} \times \frac{\pi^2}{6} \leq e^{\gamma} \times \ln\ln(2^{19} \times n)$$

where this is possible because of $3 \nmid n$ and $5 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain that

$$\ln(\frac{\pi^2}{6}) + (\ln(3+1) - \ln 3) + (\ln(5+1) - \ln 5) + \sum_{j=i}^{m}(\ln(q_j + 1) - \ln q_j) \leq$$

$$\gamma + \ln\ln\ln(2^{19} \times n).$$

From the reference [2], we note that

$$\ln(p_1 + 1) - \ln p_1 = \int_{p_1}^{p_1+1} \frac{dt}{t} < \frac{1}{p_1}.$$

In addition, note also that $\ln(\frac{\pi^2}{6}) < \frac{1}{2}$. It is enough to prove that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{q_1} + \ldots + \frac{1}{q_m} \leq \sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln\ln\ln(2^{19} \times n)$$

where $p \leq q_m$ means all the prime lesser than or equal to q_m. However, we know that

$$\gamma + \ln\ln q_m < \gamma + \ln\ln\ln(2^{19} \times n)$$

since $q_m < \ln(2^{19} \times n)$ and therefore, we would only need to prove that

$$\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln\ln q_m$$

which is true according to the Lemma 2.1 from the article reference [2]. In this way, we finally show the Theorem is indeed satisfied. □

Theorem 2.5. *Given a natural number* $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ *such that* $a_1, a_2, a_3 \geq 0$ *are integers, then the Robin's inequality is true for* n.

Proof. Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, we need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that would be the same as

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < e^{\gamma} \times \ln \ln n \tag{2.9}$$

according to Theorem 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln \ln(5040) \approx 3.81.$$

However, we know for $n > 5040$, we have that

$$e^{\gamma} \times \ln \ln(5040) < e^{\gamma} \times \ln \ln n$$

and thus, the proof is completed. □

Theorem 2.6. *The Robin's inequality is true for every natural number* $n > 5040$ *when* $15 \nmid n$. *More specifically: every counterexample should be divisible by* $2^{20} \times 3^{13} \times 5^{8} \times k_1$ *or either* $2^{20} \times 3^{13} \times k_2$ *or* $2^{20} \times 5^{8} \times k_3$, *where* $k_1, k_2, k_3 > 1$, $2 \nmid k_1$, $3 \nmid k_1$, $5 \nmid k_1$, $2 \nmid k_2$, $3 \nmid k_2$, $2 \nmid k_3$ *and* $5 \nmid k_3$.

Proof. Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we will check the Robin's inequality for n. We know this true when the greatest prime divisor of n is lesser than or equal to 5 according to Theorem 2.5. Another case is when $3 \nmid n$ and $5 \nmid n$. We need to prove the inequality (2.9) for that case. In addition, the inequality (2.9) would be true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i} < e^{\gamma} \times \ln \ln n$$

according to the Theorem 2.2. Using the properties of the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} < e^{\gamma} \times \ln \ln n$$

where $n' = q_1 \times \ldots \times q_m$ is the squarefree representation of n. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [2]. Hence, we need to prove when $2 \mid n'$. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $2^k \mid n$ for $1 \leq k \leq 19$ [4]. Consequently, we only need to prove that for all $n > 5040$ such that $2^{20} \mid n$ and thus, we have that

$$e^{\gamma} \times n' \times \ln \ln(2^{19} \times \frac{n'}{2}) < e^{\gamma} \times n' \times \ln \ln n$$

because of $2^{19} \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove that

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \ln \ln(2^{19} \times \frac{n'}{2}).$$

According to the equation (2.6) and $2 \mid n'$, we have that

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \ln \ln(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \ln \ln(2^{19} \times \frac{n'}{2})$$

which is true according to the Theorem 2.4. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $3^i \mid n$ and $5^j \mid n$ for $1 \leq i \leq 12$ and $1 \leq j \leq 7$ [4]. To sum up, we have finally proved this result. $\square$

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CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
E-mail address: `vega.frank@gmail.com`