Two Major Conjectures on Prime Numbers

Frank Vega

January 8, 2024

TWO MAJOR CONJECTURES ON PRIME NUMBERS

FRANK VEGA

To my mother

Abstract. Let $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $N_n = 2 \cdot \ldots \cdot q_n$ be the primorial of order n. We state that if the inequality $R(N_{n+1}) < R(N_n)$ holds for all primes q_n (greater than some threshold), then the Riemann hypothesis is true and the Cramér's conjecture is false. In this note, we prove that the previous inequality always holds for all sufficiently large primes q_n.

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. The hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, there have been several developments that have brought us closer to a proof of the Riemann hypothesis. There are many approaches to the Riemann hypothesis based on analytic number theory, algebraic geometry, non-commutative geometry, etc.

The Riemann zeta function $\zeta(s)$ is a function under the domain of complex numbers. It has zeros at the negative even integers: These are called the trivial zeros. The zeta function is also zero for other values of s, which are called nontrivial zeros. The Riemann hypothesis is concerned with the locations of these nontrivial zeros. Bernhard Riemann conjectured that the real part of every nontrivial zero of the Riemann zeta function is $\frac{1}{2}$.

Date: January 8, 2024.

2020 *Mathematics Subject Classification.* Primary 11M26; Secondary 11A41, 11A25.

Key words and phrases. Riemann hypothesis, Cramér's conjecture, prime numbers, Riemann zeta function, Chebyshev function.

Many thanks to Patrick Solé, Michel Planat and Yusnier Viera for their support.

The Riemann hypothesis's importance remains from its deep connection to the distribution of prime numbers, which are essential in many computational and theoretical aspects of mathematics. Understanding the distribution of prime numbers is crucial for developing efficient algorithms and improving our understanding of the fundamental structure of numbers. Besides, the Riemann hypothesis stands as a testament to the power and allure of mathematical inquiry. It challenges our understanding of the fundamental structure of numbers, inspiring mathematicians to push the boundaries of their field and seek ever deeper insights into the universe of mathematics. Indeed, the Riemann hypothesis has far-reaching implications for mathematics, with potential applications in cryptography, number theory, and even particle physics. Certainly, a proof of the hypothesis would not only provide a profound insight into the nature of prime numbers but also open up new avenues of research in various mathematical fields.

A prime gap is the difference between two successive prime numbers. The nth prime gap is the difference between the (n + 1)st and the nth prime numbers, i.e. $q_{n+1} - q_n$. The Cramér's conjecture states that $q_{n+1} - q_n = O((\log q_n)^2)$, where O is big O notation and log is the natural logarithm. Nowadays, it widely believed that Cramér's conjecture is false.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x.

Proposition 1.1. *We have* [9, pp. 1]*:*

$$\theta(x) \sim x \quad as \quad (x \to \infty).$$

We know the following inequalities:

Proposition 1.2. *For $r \geq 0$ and $-1 \leq x < \frac{1}{r}$* [6, pp. 1]*:*

$$(1+x)^r \leq \frac{1}{1 - r \cdot x}.$$

Proposition 1.3. *For $x > -1$* [6, pp. 1]*:*

$$\log(1+x) \leq x.$$

Proposition 1.4. *For $x \geq -1$ and $r > 1$* [6, pp. 1]*:*

$$(1+x)^r \geq 1 + r \cdot x.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.5. *We define* [1, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number). By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\begin{aligned}\gamma &= \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) \\ &= \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx.\end{aligned}$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Franz Mertens discovered some important results about the constant B (1874) [7].

Proposition 1.6. *Mertens' second theorem is*

$$\lim_{n \to \infty} \left(\sum_{q \leq n} \frac{1}{q} - \log \log n - B \right) = 0,$$

where $B \approx 0.26149$ is the Meissel-Mertens constant [7].

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n.

Definition 1.7. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) \geq \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

A natural number N_n is called a primorial number of order n precisely when,

$$N_n = \prod_{k=1}^{n} q_k.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) \geq \frac{e^{\gamma}}{\zeta(2)}$ is satisfied.

Proposition 1.8. *Unconditionally on Riemann hypothesis, we know that* [10, Proposition 3 pp. 3]*:*

$$\lim_{n \to \infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 1.9. *The inequality $R(N_n) > R(N_{n+1})$ is violated for infinitely many n's under the assumption that the Cramér's conjecture is true* [3, Proposition 4 pp. 5], [3, Proposition 7 pp. 7]*.*

Proposition 1.10. *For all prime numbers $q_n > 5$* [2, Theorem 1.1 pp. 358]*:*

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) < e^{\gamma} \cdot \log \theta(q_n).$$

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [4]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [5]:

$$f(x) = \Omega_R(g(x)) \textit{ as } x \to \infty \textit{ if } \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$
$$f(x) = \Omega_L(g(x)) \textit{ as } x \to \infty \textit{ if } \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_{\pm}(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as:

$$f(x) = \Omega_+(g(x)) \textit{ if } \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [8, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Finally, we have the Nicolas Theorem:

Proposition 1.11. *If the Riemann hypothesis is false then there exists a real b with $0 < b < \frac{1}{2}$ such that, as $x \to \infty$* [8, Theorem 3 (c) pp. 376]*:*

$$\log f(x) = \Omega_{\pm}(x^{-b}).$$

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there exist some implications in case of the Riemann hypothesis could be false. The following is a key Lemma.

Lemma 2.1. *If the Riemann hypothesis is false, then there exist infinitely many prime numbers q_n such that* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. The function g is defined as [10, Theorem 4.2 pp. 5]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We claim that $\mathsf{Dedekind}(q_n)$ fails whenever there exists some real number $x_0 \geq 5$ for which $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ and q_n is the greatest prime number such that $q_n \leq x_0$. It was proven the following bound [10, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}.$$

By Proposition 1.11, if the Riemann hypothesis is false, then there is a real number $0 < b < \frac{1}{2}$ such that there exist infinitely many numbers x for which $\log f(x) = \Omega_+(x^{-b})$. Actually Nicolas proved that $\log f(x) = \Omega_\pm(x^{-b})$, but we only need to use the notation Ω_+ under the domain of the real numbers. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is also $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible values of $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies that

$$\forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Note that, the value of k is not necessary in the statement above. In this way, if the Riemann hypothesis is false, then there exist infinitely many wide apart numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{1}{\sqrt{x_0}} > \frac{2}{x_0}$ for $x_0 \geq 5$, then it would be infinitely many wide apart real numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some real number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. The reason is because of the equality of the following terms:

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

3. New Criterion

This is a new Criterion for the Riemann hypothesis.

Lemma 3.1. *The Riemann hypothesis is true whenever for each large enough prime number q_n, there exists another prime $q_{n'} > q_n$ such that*

$$R(N_{n'}) \leq R(N_n).$$

Proof. By Lemma 2.1, if the Riemann hypothesis is false and the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for each large enough prime number q_n, then there exists an infinite subsequence of natural numbers n_i such that

$$R(N_{n_{i+1}}) \leq R(N_{n_i}),$$

$q_{n_{i+1}} > q_{n_i}$ and $\mathsf{Dedekind}(q_{n_i})$ fails. By Proposition 1.8, this is a contradiction with the fact that

$$\liminf_{n\to\infty} R(N_n) = \lim_{n\to\infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

By definition of the limit inferior for any positive real number ε, only a finite number of elements of $R(N_n)$ are less than $\frac{e^{\gamma}}{\zeta(2)} - \varepsilon$. This contradicts the existence of such previous infinite subsequence and thus, the Riemann hypothesis must be true. □

4. Main Insight

This is the main insight.

Theorem 4.1. *The inequality $R(N_n) > R(N_{n+1})$ holds for all primes q_n (greater than some threshold).*

Proof. For all primes q_n (greater than some threshold), we need to prove that the inequality

$$R(N_{n'}) < R(N_n)$$

is satisfied for some prime $q_{n'} > q_n$ and $n' = n + 1$. That is the same as

$$\frac{\prod_{q\leq q_{n'}} \left(1 + \frac{1}{q}\right)}{\log \theta(q_{n'})} < \frac{\prod_{q\leq q_n} \left(1 + \frac{1}{q}\right)}{\log \theta(q_n)}$$

and

$$\frac{\prod_{q\leq q_{n'}} \left(1 + \frac{1}{q}\right)}{\prod_{q\leq q_n} \left(1 + \frac{1}{q}\right)} < \frac{\log \theta(q_{n'})}{\log \theta(q_n)}$$

which is

$$\log\log\theta(q_{n'}) > \log\log\theta(q_n) + \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right)$$

after of applying the logarithm to the both sides and distributing the terms. That is equivalent to

$$1 > \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})} + \frac{\sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right)}{\log\log\theta(q_{n'})}$$

after dividing both sides by $\log\log\theta(q_{n'})$. This is possible because of the prime number $q_{n'}$ could be large enough and thus, the real number $\log\log\theta(q_{n'})$ would be greater than 0. We can apply the exponentiation to the both sides in order to obtain that

$$e > \exp\left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})}\right) \cdot \left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}}.$$

For large enough prime $q_{n'}$, we have

$$e = (\log\theta(q_{n'}))^{\frac{1}{\log\log\theta(q_{n'})}}$$

since $e = x^{\frac{1}{\log x}}$ for $x > 0$. Hence, it is enough to show that

$$\log\theta(q_{n'}) > \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right).$$

That is equal to

$$e^{\gamma} \cdot \log\theta(q_{n'}) > e^{\gamma} \cdot \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right).$$

By Proposition 1.10, we know that

$$e^{\gamma} \cdot \log\theta(q_{n'}) > \prod_{q \leq q_{n'}} \left(1 + \frac{1}{q}\right).$$

So, we deduce that

$$1 > e^{\gamma} \cdot \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

which is trivially true since

$$\lim_{n \to \infty} \left(e^{\gamma} \cdot \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}\right) = 0.$$

This is because of

$$(\log\theta(q_n))^{-1} > \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}.$$

We can check that

$$\lim_{n\to\infty}\left(e^{\gamma}\cdot(\log q_n)^{-1}\right)=0$$

is true since

$$\theta(q_n)\sim q_n \quad as \quad (n\to\infty)$$

by Proposition 1.1. Actually, the point here is the statement

$$(\log\theta(q_n))^{-1}>\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)^{-1}$$

should be true for large enough n which is equal to say that $R(N_n)>1$ holds indeed. By Proposition 1.8, there exists a value of m_0 so that for all natural numbers $m\geq m_0$

$$\liminf_{m\to\infty}R(N_m)-\epsilon=\frac{e^{\gamma}}{\zeta(2)}-\epsilon<R(N_m)<\frac{e^{\gamma}}{\zeta(2)}+\epsilon=\limsup_{m\to\infty}R(N_m)+\epsilon$$

for every arbitrary and absolute value $\epsilon>0$ by definition of limit superior and inferior due to

$$\liminf_{m\to\infty}R(N_m)=\limsup_{m\to\infty}R(N_m)=\lim_{m\to\infty}R(N_m).$$

In this way, it should exist some value of n_0 so that for all natural numbers $n\geq n_0$ we obtain that $R(N_n)>1$ since $\frac{e^{\gamma}}{\zeta(2)}>1$. We would have

$$1+\epsilon_1=\exp\left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})}\right)$$

and

$$e\cdot(1-\epsilon_2)=\left(\prod_{q_n<q\leq q_{n'}}\left(1+\frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}}.$$

We only need to prove that

$$e>(1+\epsilon_1)\cdot e\cdot(1-\epsilon_2)$$

which is the same as

$$\epsilon_2>\frac{\epsilon_1}{\epsilon_1+1}.$$

In addition, we can see that

$$1-e^{-1}\cdot\left(\prod_{q_n<q\leq q_{n'}}\left(1+\frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}}=\epsilon_2.$$

We have

$$\left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}} = \left(1 + \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right) - 1\right)^{\frac{1}{\log\log\theta(q_{n'})}}$$
$$\leq \frac{1}{1 - \frac{\left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right) - 1\right)}{\log\log\theta(q_{n'})}}$$
$$= \frac{\log\log\theta(q_{n'})}{\log\log\theta(q_{n'}) + 1 - \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)}$$

by Proposition 1.2, since

$$-1 \leq \left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right) - 1\right) < \log\log\theta(q_{n'})$$

due to q_n and $q_{n'}$ are large enough. We can show the inequality

$$\left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right) - 1\right) < \log\log\theta(q_{n'})$$

could hold for a large enough prime q_n as well. Indeed, we are able to show that is equal to

$$\left(\sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right) - \frac{1}{q}\right) < -\left(\sum_{q_n < q \leq q_{n'}} \frac{1}{q}\right) + \log\log\log(\theta(q_{n'}))^e$$

after of applying the logarithm and adding the term

$$-\left(\sum_{q_n < q \leq q_{n'}} \frac{1}{q}\right)$$

to the both sides. By Proposition 1.3, we verify that

$$0 \geq \left(\sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right) - \frac{1}{q}\right).$$

By Proposition 1.6, if we get any large enough prime number q_n such that

$$\log\log\log(\theta(q_{n'}))^e \geq \left(\sum_{q_n < q \leq q_{n'}} \frac{1}{q}\right) \approx (\log\log q_{n'} - \log\log q_n)$$

which is

$$(q_{n'})^{\frac{1}{1+\log\log\theta(q_{n'})}} \lessapprox q_n,$$

then this could be quite good for supporting our claim. As a consequence, we obtain that

$$1 - \frac{e^{-1} \cdot \log \log \theta(q_{n'})}{\log \log \theta(q_{n'}) + 1 - \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)} < \epsilon_2.$$

Putting all together, we show that

$$1 - \frac{e^{-1} \cdot \log \log \theta(q_{n'})}{\log \log \theta(q_{n'}) + 1 - \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)} \geq \frac{\epsilon_1}{\epsilon_1 + 1}.$$

That is equivalent to say that

$$(1 - e^{-1}) \cdot \log \log \theta(q_{n'}) + 1 - \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)$$
$$\geq \frac{\epsilon_1}{\epsilon_1 + 1} \cdot \left(\log \log \theta(q_{n'}) + 1 - \prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)\right)$$

could be satisfied. However, the previous inequality truly holds since

$$(1 - e^{-1}) > \frac{\epsilon_1}{\epsilon_1 + 1} = 1 - \frac{1}{\epsilon_1 + 1}$$

is satisfied. Certainly, that would be equivalent to say that

$$e > \epsilon_1 + 1$$

which is

$$e > \exp\left(\frac{\log \log \theta(q_n)}{\log \log \theta(q_{n'})}\right)$$

since

$$\epsilon_1 = \exp\left(\frac{\log \log \theta(q_n)}{\log \log \theta(q_{n'})}\right) - 1.$$

Based on our pre-condition $n' = n + 1$, then this implies that

$$(1 - e^{-1}) \cdot \log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}} \geq \frac{\epsilon_1}{\epsilon_1 + 1} \cdot \left(\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}\right)$$

holds. Indeed, that would be the same as

$$\frac{(1 - e^{-1}) \cdot \log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}} \geq \frac{\epsilon_1}{\epsilon_1 + 1}$$

that is

$$-\frac{e^{-1} \cdot \log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}} \geq -\frac{1}{\epsilon_1 + 1}$$

and

$$\frac{q_{n+1} \cdot \log \log \theta(q_{n+1})}{q_{n+1} \cdot \log \log \theta(q_{n+1}) - 1} \leq \exp\left(1 - \frac{\log \log \theta(q_n)}{\log \log \theta(q_{n+1})}\right)$$

which could be true according to

$$\exp\left(1-\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right) \geq \exp\left(\frac{1}{q_{n+1}\cdot\log\log\theta(q_{n+1})-1}\right)$$
$$\geq 1+\frac{1}{q_{n+1}\cdot\log\log\theta(q_{n+1})-1}$$
$$=\frac{q_{n+1}\cdot\log\log\theta(q_{n+1})}{q_{n+1}\cdot\log\log\theta(q_{n+1})-1}$$

by Proposition 1.3 for large enough prime q_n. Indeed, that is equal to

$$\log\left(\frac{q_{n+1}\cdot\log\log\theta(q_{n+1})}{q_{n+1}\cdot\log\log\theta(q_{n+1})-1}\right)\leq 1-\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}$$

which is

$$\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\leq \log\left(\frac{e\cdot(q_{n+1}\cdot\log\log\theta(q_{n+1})-1)}{q_{n+1}\cdot\log\log\theta(q_{n+1})}\right)$$

after of applying the logarithm to the both sides and distributing the terms. That would be

$$(\log\theta(q_n))^{\frac{1}{\log\log\theta(q_{n+1})}}\leq \left(\frac{e\cdot(q_{n+1}\cdot\log\log\theta(q_{n+1})-1)}{q_{n+1}\cdot\log\log\theta(q_{n+1})}\right)$$

after of doing a simple exponentiation. We know that

$$(\log\theta(q_n))^{\frac{1}{\log\log\theta(q_{n+1})}} < (\log\theta(q_{n+1}))^{\frac{1}{\log\log\theta(q_{n+1})}} = e.$$

For that reason, we deduce that

$$\left(\frac{\log\theta(q_n)}{\log\theta(q_{n+1})}\right)^{\frac{1}{\log\log\theta(q_{n+1})}}\leq 1-\left(\frac{1}{q_{n+1}\cdot\log\log\theta(q_{n+1})}\right)$$

and

$$\frac{\log\theta(q_n)}{\log\theta(q_{n+1})}\leq\left(1-\left(\frac{1}{q_{n+1}\cdot\log\log\theta(q_{n+1})}\right)\right)^{\log\log\theta(q_{n+1})}.$$

By Proposition 1.4, we have

$$\frac{\log\theta(q_n)}{\log\theta(q_{n+1})}\leq 1-\frac{1}{q_{n+1}}$$

since $-\frac{1}{q_{n+1}\cdot\log\log\theta(q_{n+1})}\geq -1$. That would be

$$\frac{\log\theta(q_{n+1})}{q_{n+1}}\leq \log\theta(q_{n+1})-\log\theta(q_n).$$

However, we have [3, pp. 4]:

$$\log\theta(q_{n+1})-\log\theta(q_n)=\log\log N_{n+1}-\log\log N_n=\log\left(1+\frac{\log q_{n+1}}{\theta(q_n)}\right)$$

and therefore,

$$\frac{\log\theta(q_{n+1})}{q_{n+1}}\leq\log\left(1+\frac{\log q_{n+1}}{\theta(q_n)}\right)$$

where

$$(\theta(q_{n+1}))^{\frac{1}{q_{n+1}}} \leq \left(1 + \frac{\log q_{n+1}}{\theta(q_n)}\right)$$

which is

$$\theta(q_n) \leq (\theta(q_{n+1}))^{1-\frac{1}{q_{n+1}}}$$

that is trivially true for large enough prime q_n since

$$\left(1 - \frac{1}{q_{n+1}}\right) \to 1 \quad as \quad (n \to \infty)$$

and

$$\lim_{n\to\infty} (\theta(q_{n+1}) - \theta(q_n)) = +\infty.$$

Now, the proof is done. □

5. Main Theorem

This is the main theorem.

Theorem 5.1. *The Riemann hypothesis is true and the Cramér's conjecture is false.*

Proof. By Lemma 3.1, the Riemann hypothesis is true if for all primes q_n (greater than some threshold), the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for some prime $q_{n'} > q_n$. Therefore, the Riemann hypothesis is true by Theorem 4.1. We also know the the Cramér's conjecture is false as a consequence of Proposition 1.9 and Theorem 4.1. □

6. Conclusion

Practical uses of the Riemann hypothesis include many propositions that are considered to be true under the assumption of the Riemann hypothesis and some of them that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. In general, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas.

References

1. Raymond Ayoub, *Euler and the Zeta Function*, The American Mathematical Monthly **81** (1974), no. 10, 1067–1086.
2. YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé, *On Robin's criterion for the Riemann hypothesis*, Journal de Théorie des Nombres de Bordeaux **19** (2007), no. 2, 357–372.
3. YoungJu Choie, Michel Planat, and Patrick Solé, *On Nicolas criterion for the Riemann hypothesis*, arXiv preprint arXiv:1012.3613 (2010).

4. Godfrey Harold Hardy and John Edensor Littlewood, *Some problems of diophantine approximation: Part II. The trigonometrical series associated with the elliptic ϑ-functions*, Acta mathematica **37** (1914), no. 1, 193–239.
5. ———, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Mathematica **41** (1916), 119–196.
6. László Kozma, *Useful Inequalities. Kozma's Homepage, Useful inequalities cheat sheet*, `http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf`, 2011–2024, Accessed 8 January 2024.
7. Franz Mertens, *Ein Beitrag zur analytischen Zahlentheorie.*, J. reine angew. Math. **1874** (1874), no. 78, 46–62.
8. Jean-Louis Nicolas, *Petites valeurs de la fonction d'Euler*, Journal of Number Theory **17** (1983), no. 3, 375–388.
9. David J Platt and Timothy S Trudgian, *On the first sign change of $\theta(x) - x$*, Mathematics of Computation **85** (2016), no. 299, 1539–1547.
10. Patrick Solé and Michel Planat, *Extreme values of the Dedekind ψ function*, Journal of Combinatorics and Number Theory **3** (2011), no. 1, 33–38.

NataSquad, 10 rue de la Paix 75002 Paris, France
Email address: `vega.frank@gmail.com`