Consensus in Multi-Agent System with Switching Topology

Amol Patil and Gautam Shah

July 3, 2024

Consensus in Multi-Agent System With Switching Topology

Amol Patil and Gautam Shah

Abstract— **This article discusses mathematical framework of development of consensus algorithm for multi-agent systems using switching topology. Consensus equation is derived in discrete time domain using Perron Frobenius theory. Discrete time consensus equation is depends upon underline topology of multi-agents. For achieving consensus two cases are considered 1) Directed networks with fixed topology 2) Directed networks with switching topology. Convergence analysis of consensus algorithm for switching topology is compared with respect to fixed topology. A distinctive feature of this article is to address consensus problems for networks with directed information flow. Simulations are provided that demonstrate the effectiveness of our theoretical results**

Keywords : Multi-agent system (MAS), Consensus algorithm, Communication graph, Algebraic graph theory

I. INTRODUCTION

Control of multi-agent systems (MAS) is an active area of research since two decades [1]. Control of MAS system is generally solved by two approaches: a centralized approach and a distributed approach. Distributed approach is more widely used and more promising because of unavoidable physical constraints such as short wireless communication range, limited sensing region, narrow bandwidth, large sizes of vehicles and complex dynamics of each agent [2]. These features are difficult to manage and control that's why distributed control of MAS becomes promising area of research. Many survey papers [3], [4] and [5] explains recent progress on development of coordination and control of MAS.

Consensus is the fundamental and important problem in MAS hence some consensus algorithms were studied under various information flow constraints [6], [7], [8], [9] and [10]. In addition, there are some recent reviews and progress reports given in the monographs [11], [13]. This article describes design of consensus algorithm in MAS for fixed as well as switching topology using discrete Perron Frobenius theory. It also provides convergence analysis of consensus algorithm for switching topology with comparison to fixed topology

Notations and Definitions: The notations used in this paper are standard: Capital bold face letters are used for representing the matrix and the elements of a matrix are represented by small case letters. $\mathbb{R}^{n\times n}$ denote the set of $n\times n$ real matrices. G_N represents the communication graph of dimension N. λ, $\underline{1}$ and w_1 denotes the eigenvalues, right eigen vectors and left eigen vectors of Frobenius matrix **F** respectively. Matrices **F**, **D** and **A** denote Frobenius matrix, diagonal matrix and connectivity matrix respectively

The organization of paper is as follows. Section II presents preliminaries on algebraic graph theory and graph matrices. Section III describes problem formulation. Simulation result is explained in section IV and conclusion and future scope is stated in section V

II. PRELIMINARIES

This section explains algebraic graph theory a mathematical tool require for construction of communication graph for the multi-agent system (MAS). The behaviors and connections of the MAS are possible through edges of a communication graph, hence covering graph theory becomes essential. Information between agents is either unidirectional or bidirectional. Graph matrices associated with communication graph is introduced later in section III where focus is shifted to connectivity and Laplacian matrices which are commonly used in the MAS.

Communication graph is defined as a set of agents and a edge set of unordered pair of nodes connected over the graph. Mathematically a communication graph is represented as $G_N = (V, E)$, where $V = \{v_1, v_2, \ldots, v_N\}$ represents set of agents, and E denoted as (v_i, v_j) signifies the connection between the agents starting from i and terminating at j [12]. The in-degree of a node denotes the number of edges that have v_i as their head, and the out-degree of a node means how many edges has v_i as their tail. The in-degree of a agent in graph equals to the number of neighbors of that node. Knowing the number of neighbors of node i, i.e. $N_i = v_j : (v_i, v_j) \in E$ is crucial in consensus algorithm. The information sharing between the each agents for underlying communication graph is distributed in the sense that each agent communicates with its neighbors locally so as to achieve the global objective. Total number of agents present in a communication graph are N. Information exchange among agents is modeled by directed and undirected graph. The edge weights in a graph are strictly positive. For a graph if there are directed path from v_i to v_j and from v_j to v_i then the graph is bidirectional or undirected graph. Every Undirected graph is balanced graph. For a graph if there is directed path from

Amol Patil is with Phd student of Electronics and Telecommunication Engineering, St. Francis Institute of Technology, University of Mumbai, Borivali Mumbai-400 103 agpatil@acpce.ac.in

Gautam Shah is with faculty of Electronics and Telecommunication Engineering, St. Francis Institute of Technology, University of Mumbai, Borivali Mumbai-400 103 gautamshah@sfit.ac.in

v_i to v_j and not from v_j to v_i then the graph is directed graph, if such graph does not maintain degree at each agent is known as unbalanced graph. For various other types of graph refer [12] and [13]

Communication graph has at least one directed spanning tree in which all the agents of the graph and are reachable from root agent by following the edge arrows. A graph may have multiple spanning trees but existence of one spanning tree is enough for achieving the consensus. If the graph is strongly connected then all the nodes in the graph are root nodes. More about algebraic graph theory is given in [12] and [13]

Communication graph properties can be studied by examining the properties of certain matrices associated with the graph. For the given communication graph the degree of each node is denoted by $d(v_i)$ and it equal to number of incoming branches at node v_i. Diagonal matrix denoted by $\mathbf{D}$ with diagonal elements equal to the indegree of each node [13]. Connectivity matrix $\mathbf{A}$ have special importance in a graph since it consists of information regarding the agents and their interconnections [13]. The adjacency matrix $\mathbf{A}$ of a graph is square matrix defined as

$$\mathbf{A} = \begin{cases} 1 & (v_i v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

For balanced graph the matrix $\mathbf{A}$ is symmetric and for unbalanced graph it is non symmetric. Graph Laplacian matrix $\mathbf{L}$ play a important role for achieving consensus in multi-agent system and is defined by $\mathbf{L} = \mathbf{D} - \mathbf{A}$.

Many properties of a graph may be studied in terms of the matrix $\mathbf{L}$. Here it is vital to denote some of the important characteristics of the matrix $\mathbf{L}$ for the given graph that are applicable to the multi-agent system obtained from [**?**].

- $\mathbf{L}$ is symmetric (respectively non symmetric) if G_N is balanced (respectively if G_N is unbalanced)
- $\mathbf{L}$ is positive semi-definite (respectively positive definite) if G_N is balanced (repectively if G_N is unbalanced)
- $\mathbf{L}$ consist of real eigen values (respectively complex eigen values) if G_N is balanced (repectively if G_N is unbalanced)
- Row sum of $\mathbf{L}$ for each row is equal to zero

Eigen structure of the matrix $\mathbf{L}$ plays the crucial role for analyzing the convergence of the consensus algorithm at the steady state. One of the eigen value of the matrix $\mathbf{L}$ is simple and located at origin then system is said to be critically stable

III. PROBLEM FORMULATION

This section explains the mathematical formulation of discrete time consensus algorithm. For deriving the discrete time consensus equation each agent in multi-agent system is associated with following discrete time state space equation

$$x_i(k+1) = x_i(k) + \mu_i(k) \tag{1}$$

with k is present time instance and $k+1$ is the future time instance. x_i is the state information of agent i and μ_i is the control input of agent i and $x_i, \mu_i \in \mathbb{R}$. The discrete time consensus equation is derived using Perron discrete time system in section III-A

A. Normalized Control Protocol For Discrete-Time Consensus

Consider the following normalized control input for each agent

$$\mu_i(k) = \frac{1}{1+d_i} \sum_{j \in N_i} a_{ij} \left[x_j(k) - x_i(k) \right] \tag{2}$$

where a_{ij} are elements of matrix $\mathbf{A}$ and d_i is the in-degree of agent i. Substituting equation (2) in equation (1), we get

$$x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \sum_{j \in N_i} a_{ij} \left[x_j(k) - x_i(k) \right] \tag{3}$$

$$\begin{aligned} x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \Bigg(& - x_i(k) \sum_{j \in N_i} a_{ij} \\ & + \sum_{j \in N_i} a_{ij} x_j(k) \Bigg) \end{aligned} \tag{4}$$

But $\sum_{j \in N_i} a_{ij} = d_i$ and if $j = 1....N$ then equation (4) reduces to

$$\begin{aligned} x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \Bigg(& - x_i(k) d_i \\ & + \begin{bmatrix} a_{i1} & \dots & a_{iN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \Bigg) \end{aligned} \tag{5}$$

Matrix form representation of equation (5) is given by

$$x(k+1) = \mathbf{I}x(k) + \frac{\mathbf{I}}{\mathbf{I}+\mathbf{D}} \Bigg(- x(k)\mathbf{D} + \mathbf{A}x(k) \Bigg) \tag{6}$$

Here matrix $\mathbf{I}$ is identity matrix equivalent to 1 in scalar, taking $x(k)$ common equation (6) reduced to

$$x(k+1) = \Bigg[\mathbf{I} + \frac{\mathbf{I}}{\mathbf{I}+\mathbf{D}} \Big[- \big(\mathbf{D} - \mathbf{A} \big) \Big] \Bigg] x(k) \tag{7}$$

simplifying equation (7) finally we get discrete time consensus equation as

$$x(k+1) = (\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A})x(k) \equiv \mathbf{F}x(k) \tag{8}$$

In case of switching toplogy equation (8) is modified as

$$x(k+1) = (\mathbf{I}+\mathbf{D})^{-1}(\mathbf{I}+\mathbf{A})x(k) \equiv \mathbf{F(k)}x(k) \tag{9}$$

with $x = [x_1......x_N]^T \in \mathbb{R}^N$. The matrix $\mathbf{F}$ is referred as the Frobenius matrix. Laplacian matrix **L =D-A** has one eigenvalue at origin of complex s-plane and the remaining eigenvalues in the right half of the complex s-plane, makes system unstable. To make it stable poles of the system brought to the left of complex s-plane by assigning negative sign to matrix **L= D-A** refer equation (7) and according to Gershgorin circle criterion in the normalized Laplacian matrix $(\textbf{I+D})^{-1}(\textbf{D-A})$ all eigen values are found on the right-hand side of the complex z-plane within a disk centered at $\frac{d_{max}}{1+d_{max}}$ with the same size radius [13]. The shaded region of the z-plane is shown in Fig. 1 where all eigenvalue reside. If $\frac{d_{max}}{1+d_{max}} < 1$ then this shaded region is always inside the unit circle. Thus **F** has a simple eigenvalue of $\lambda_1 = 1$ and the remaining eigen values are strictly inside the unit circle. This result make equation (8) is marginally stable and of Type 1, and state approaches a steady-state value is reached [13]

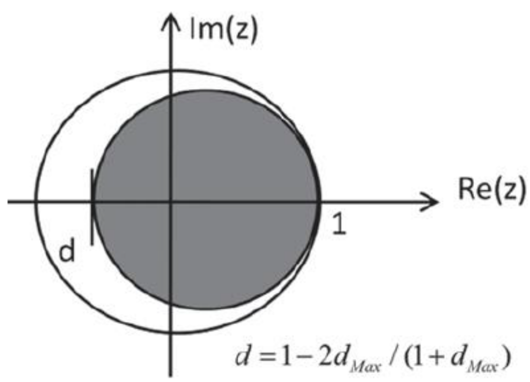


Fig. 1: Region of eigenvalues of Frobenius matrix F

Since Laplacian matrix **L** has row sums of zero, then matrix **F** has row sums of 1, and so is a row stochastic matrix. That is

$$\mathbf{F}\underline{1} = \underline{1} \tag{10}$$

and $\underline{1}$ is the right eigenvector for the eigen value $\lambda_1 = 1$. Let w_1 be a left eigenvector of matrix **L** for $\lambda_1 = 0$. Then w_1 is also the left eigenvector of matrix **F** for $\lambda_1 = 1$

B. STEADY STATE ANALYSIS

This section explains the convergence of discrete time consensus algorithm at steady state. If the system (8) reaches to steady state, then

$$x_{ss} = \mathbf{F}x_{ss} \tag{11}$$

If the given network topology has a directed spanning tree then only possible solution is $x_{ss} = c\underline{1}$ for some $c > 0$. Then the consensus achieved such that $x_i = x_j = c \ \forall \ i, j$. Let $w_1 = [p_1, ..., p_N]^T$ be a left eigenvector of **F** for $\lambda_1 = 1$, then

$$w_1^T x(k+1) = w_1^T \mathbf{F} x(k) = w_1^T x(k) \tag{12}$$

So that the quantity

$$\bar{x} = w_1^T x = [p_1, ..., p_N]^T \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_N \end{bmatrix} = \sum_i p_i x_i \tag{13}$$

is invariant. This quantity $\bar{x} = \sum_i p_i x_i$ is the constant of motion. So $\sum_i p_i x_i(0) = \sum_i p_i x_i(k) \ \forall \ k$. Therefore if the network topology has a spanning tree, at steady state one can reached the consensus so that $x_i = x_j = c \ \forall \ i, j$ where the consensus value is given by

$$c = \frac{\sum_i p_i x_i(0)}{\sum_i p_i} \tag{14}$$

IV. SIMULATION RESULT

Consider a fixed topology shown in Fig. 2 and consider all

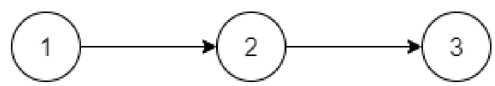


Fig. 2: Fixed Topology

edge weight a_{ij} is equal to 1 for mathematical sophistication. The Frobenius matrix **F** for fixed topology shown in Fig. 2 is given by

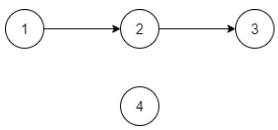
$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \tag{15}$$

Consider Switching topology is as shown in Fig. 3, and there are 3 Frobenius matrices associated with switching topology and same are determined using equation (9) which are given by **F(0)**, **F(1)** and **F(2)** and which are given below

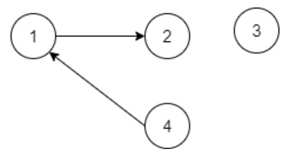
$$\mathbf{F(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{bmatrix} \tag{16}$$

$$\mathbf{F(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix} \tag{17}$$

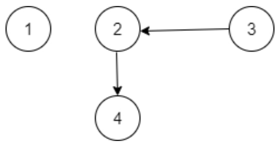
$$\mathbf{F(2)} = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$



(a) G1



(b) G2



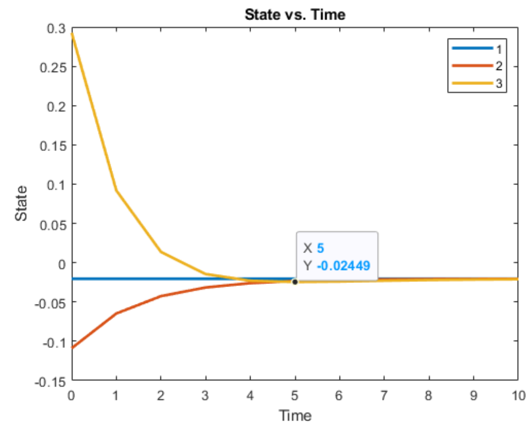
(c) G3

Fig. 3: Switching Topology

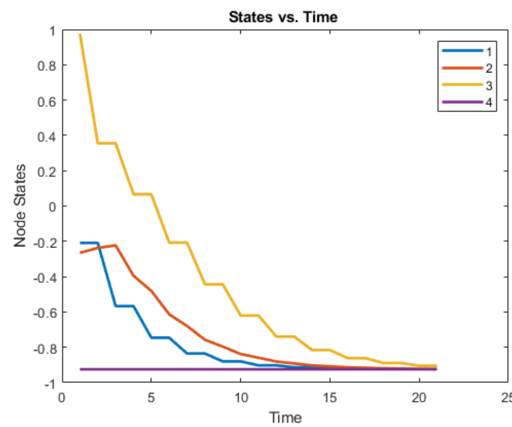
Eigen values for Frobenius matrix $\mathbf{F}$ are $\lambda_1 = 1$, $\lambda_{2,3} = 0.5$ and corresponding left eigenvectors for Frobenius matrix $\mathbf{F}$ are $p_1 = 1$, $p_2 = 0$ and $p_3 = 0$. Here initial state information for agent 1, agent 2 and agent 3 are $x_1(0) = -0.02$, $x_2(0) = -0.1$ and $x_3(0) = -0.3$ respectively. using equation (14) consensus value determined as -0.02 refer Fig. 4a However for switching topology Frobenius matrices are $\mathbf{F(0)}$, $\mathbf{F(1)}$ and $\mathbf{F(2)}$. For $\mathbf{F(0)}$ the eigen values are $\lambda_1 = 1$, $\lambda_{2,3} = 0.5$ and $\lambda_4 = 0$ and corresponding eigen vectors are $p_i = 1, 0, 0, 0$. For $\mathbf{F(1)}$ the eigen values are $\lambda_{1,4} = 1$, $\lambda_{2,3} = 0.5$ and corresponding eigen vectors are $p_i = 0, 0, 1, 0$. For $\mathbf{F(2)}$ the eigen values are $\lambda_{1,2} = 0.5$, $\lambda_{3,4} = 1$ and corresponding eigen vectors are $p_i = 0, 0, 1, 1$. This structure $G1$, $G2$, $G3$ repeates 21 times then using equation (14) consensus value determined as -0.9 refer Fig. 4b.

V. Conclusion

From simulation result shown in Figure 4a state information of all agents converges to consensus value of -0.02 at **5** seconds and simulation result shown in Figure 4b state information of all agents converges to consensus value at **20** seconds. Consensus value of switching topology reaches later as compare to consensus value reached by using fixed topology because there are link failure in graph G1, graph G2 and graph G3 refer Figure 3. The Frobenius matrix for switching topology is given by F(0), F(1) and F(2). Initial state information for fixed and switching topology is selected randomly and lies in between the range of $+1$ to -1



(a) Fixed Topology Plot



(b) Switching Topology Plot

Fig. 4: Consensus Convergence

References

[1] Yongcan Cao and et. al., “An Overview of Recent Progress in the Study of Distributed Multi-agent Coordination”, *IEEE Transactions on Industrial informatics*, Vol.9, no.1, pp. 427-438, 2013

[2] Fei Chen and Wei Ren, ”On the control of multi-agent systems: A survey.” *Foundations and Trends in Systems and Control*, Vol.6, No. 4, pp 339-499, 2019

[3] V Gazi, and B. Fidan, ”Coordination and control of multi-agent dynamic systems: Models and approaches.” *In International Workshop on Swarm Robotics*, pp. 71-102. Springer, Berlin, Heidelberg, 2006.

[4] R. M. Murray, ”Recent research in cooperative control of multivehicle systems” *Journal of Dynamic Systems, Measurement and Control*, Vol. 129, No. 5, pp 571-583, 2007

[5] P. Y. Chebotarev and R. P. Agaev, ”Coordination in multiagent systems and Laplacian spectra of digraphs.” *Automation and Remote Control* Vol. 70, no. 3, pp. 469-483, 2009

[6] A. Jadbabaie, J. Lin, and A.S. Morse, ”Coordination of groups of mobile autonomous agents using nearest neighbor rules”,*IEEE Trans. on Automatic Control*, 48(6):988–1001, 2003.

[7] J. A. Fax and R. M. Murray, “Information flow and cooperative control

of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, September 2004.

[8] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," IEEE Transactions on Automatic Control, vol. 49, no. 9, pp. 1520–1533, September 2004

[9] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no.5, pp. 655–661, May 2005

[10] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, January 2007

[11] C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*. World Scientific, 2007.

[12] Z. Qu, *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles*, Springer-Verlag, 2009

[13] Frank Lewis, Hongwei Zhang, Kristian Hengster-Movric and Abhijit Das. Cooperative control of multi-agent systems: optimal and adaptive design approaches. Springer Science Business Media, 2013