Novel Complex Hopfield Neural Networks: Convergence Theorems

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ABSTRACT

In this research paper, a novel proof of convergence theorem associated with a complex valued neural network based on complex signum function is proved. Also, two novel Complex Valued Neural Networks (CVNNs) are proposed. One of them is based on magnitude and phase quantization using Ceiling type activation function operating on rectangular coordinate representation of complex net contribution. The other CVNN is also based magnitude and phase quantization using Ceiling type activation function operating on polar coordinate representation of the complex net contribution. The converence theorems associated with such novel CVNNs are also proved.

1.INTRODUCTION:

Traditionally, real valued Artificial Neural Networks (ANNs) were proposed based on the McCulloch-Pitts model of neuron. In such model, the inputs, synaptic weights, threshold, output were real values numbers. N.N. Aizenberg generalized such a model of neuron with the inputs, threshold, synaptic weights, output being complex numbers. Researchers extensively investigated various aspects of such, so called Complex Valued Neural Networks (CVNNs). The author proposed an interesting model of complex valued neuron based on complex signum function as the activation function. In literature, researchers proposed a model of complex valued Hopfield neural network based on a different activation function involving phase quantization only. The author, based on complex signum function proposed an interesting complex valued Hopfield neural network and proved the associated convergence Theorem (like the convergence Theorem for real valued Hopfiled neural network). Such an ANN employs one type of magnitude quantization. Such early efforts motivated the complex valued Hopfield Neural Network proposed in this research paper based on activation function involving magnitude and phase quantization. Two interesting activation functions are proposed in this research paper. We provide an approach to prove the associated convergence Theorems which explan the associative memory function of the novel complex valued Hopfield neural networks.

This research paper is organized as follows. In Section 2, related research literature is reviewed. In Section 3, a novel proof of convergence Theorem associated with a CVNN based on magnitude quantization using complex signum function is proved. In Section 4, two novel CVNNs based on magnitude and phase quantization are proposed based on magnitude and phase quantization. The associated convergence Theorems are proved. The research paper concludes in Section 5.

2. **Review of Related Research Literature**:

Motivated by the problem of emulating the biological memory, Hopfield proposed a model of associative memory [1]. Goles and Fogelman innovated an interesting approach to prove the convergence Theorem associated with the Hopfield Associative Memory (HAM) conceived by Hopfield. Based on complex valued neurons, Muezzinoglu et.al proposed a model of Complex Hopfield Neural Network (CHNN) and applied it to image processing [3]. Such a CHNN is based on phase quantization (of net potential) related activation function. This activation function originated in [2]. The author and his team proposed a CHNN based on a type of magnitude quantization and proved the associated convergence Theorem [4]. Related Complex Valued neural networks were proposed in [5]. These early efforts motivated the author to conceive of two interesting activation functions which employ magnitude as well as phase quantization. Based on such activation function, we innovate novel CHNN's and prove the associated convergence Theorems (with early proof techniques as the foundation).

3. Magnitude Quantization: Novel Proof of Convergence Theorem:

Consider a complex valued neural network with 'L" neurons and the associated Hermitian Synaptic weight matrix, K (i.e. $K_{ij} = K_{ij}^*$, where '*" denotes the complex conjugate). Let the complex signum function be defined in the following manner: Given a complex number z= a + j b, CoSign(z) = Sign(a) + j Sign(b). The state of every neuron belongs to the set J = { 1+j1, 1-j1, -1+j 1, -1-j1 }. Thus, the state vector, an L x 1 column vector lies on the unit, symmetric complex hypercube (with all components of the vector taking values from the set J). As in the case of real valued Hopfiled Neural Network (RHNN), the proposed CHNN operates in two modes of operation with

$$\bar{V}(n) = [\, V_1(n) \;\; V_2(n) \ldots . V_L(n)]$$

as the state vector at time 'n' and the following dynamics in two modes of operation:

Serial Mode:

$$V_i(n+1) = CoSign\left(\sum_{j=1}^{L} K_{ij}\, V_j(n) - T_i\right),$$

where T_i is the threshold value at neuron 'i' for $n \geq 0$.

Fully Parallel Mode:

$$\bar{V}(n+1) = CoSign[\bar{K}\,\bar{V}(n) - \bar{T}\,]$$, for $n \geq 0$, where $\bar{T}$ is the threshold vector.

In the state space (unit complex hypercube) of such a CVNN, there are special states, called "stable states", "anti-stable states" defined in the following manner:

Stable State: $\bar{J}$ *such that* $\bar{J} = CoSign\,(\,\bar{K}\bar{J} - \bar{T}\,)$.

Anti-Stable State: $\bar{P}$ *such that* $\bar{P} = CoSign\,(\,\bar{K}\bar{P} - \bar{T}\,)$.

The non-linear dynamics of proposed CHNN is captured by the following convergence Theorem:

THEOREM 1: Consider the CHNN proposed above. Such a CHNN with all diagonal elements of the synaptic weight matrix, $\overline{K}$ being non-negative

(i) Converges to a stable state in the serial mode of operation with any initial condition lying on the complex, unit symmetric hypercube and

(ii) Converges to a stable state or a cycle of length two (2) in the fully parallel mode of operation (starting in any initial state lying on the unit symmetric hypercube).

In [ ], a proof of the above convergence Theorem was proved. We now provide a new proof of the above convergence theorem associated with the CHNN proposed. We need the following lemma which deals with the quadratic form associated with a Hermitian matrix. The following Lemma is very general and reduces the Hermitian form to the sum of two real quadratic forms:

LEMMA 1: Consider a Hermitian matrix, $\bar{B} = B_R + jB_I$, where the matrices $\{ B_R, B_I \}$ are the real and imaginary parts of $\bar{B}$ respectively. Also, let $\bar{X}$ be a vector with complex numbers as the components such that $\bar{X} = X_R + jX_I$. . The following decomposition of Hermitian form associated with $\bar{B}$ holds true:

$$\bar{X}^{*}\bar{B}\,\bar{X} \;= X_R^T B_R X_R + X_I^T B_R X_I$$

Proof: Follows from the argument first documented in [6].

Note: The above Lemma is significant in the sense that the energy function (quadratic form) associated with Hermitian matrix

NOVEL PROOF OF THEOREM 1: The proof of above convergence theorem (i.e. Theorem 1) associated with the above Complex Hopfield Neural Network (CVNN) (first proved in [4]) is based on showing that the energy function (i.e. quadratic form associated with the Hermitian synaptic weight matrix, $\overline{K}$) is non-decreasing in the serial mode of operation and converges to a stable state. But the Lemma 1 shows that the energy function reduces to the sum of real quadratic forms associated with the real-valued synaptic weight matrix , say $\overline{K}_R$. Thus, we directly invoke the proof of convergence Theorem associated with the real-valued Hopfield neural network and arrive at the desired conclusion. Thus, we have a different proof of THEOREM 1 based on Lemma 1.

4. Novel Complex Valued Hopfield Neural Networks: Magnitude and Phase Quantization:

- In the literature, researchers proposed complex valued “multistate” neural associative memories based on an interesting activation function, called “complex signum” function (employed at each neuronal node), utilizing certain phase quantization. We now provide the definition of such activation function:

Definition: Complex Signum Function based on Phase Quantization:

Let ‘S’ be a positive integer, called the RESOLUTION FACTOR and let ‘u’ be an arbitrary complex number. The complex signum function is defined as follows:

$$\text{CSIGN(u)} \triangleq \begin{cases} z^0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots for\ \ 0 \le \arg(u) < \theta_0 \\ z^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots' for\ \theta_0 \le \arg(u) < 2\theta_0 \\ \vdots \\ z^{s-2} \ldots for\ (s-2)\,\theta_0 \le \arg(u) < (s-1)\theta_0 \\ z^{s-1} \ldots\ldots\ldots\ldots,\ \ for (s-1) \le \arg(u) \le s\theta_0 \end{cases}$$

Where 'arg' denotes the argument (phase angle) of a complex number,

$\theta_{0:}$ $is\ a\ phase\ quantum\ denoted\ by\ S$: $\theta_0 = \frac{2\pi}{S}$ and

$$z\, is\ the\ corresponding\ kth\ root\ of\ one: z = e^{j\theta_0}.$$

Thus, CSIGN(u) corresponds to 'S' discrete set og complex numbers located uniformly on the unit circle in the complex plane.

In the asynchronous/serial mode of operation of such CVNN, the state of ith neuron at time 'k' i.e. v_i(k) is updated in the following manner:

$$v_i(k+1) = CSIGN\left[z^{\frac{1}{2}} \sum_{j=1}^{L} k_{ij} v_j(k) \right],$$

$$where\ z^{\frac{1}{2}} = e^{j\frac{\theta_0}{2}}.$$

The nonlinear dynamics of such a complex valued Hopfield neural network was extensively studied in [3], [4] and was applied to image processing.

Motivated by the earlier research efforts on complex valued Hopfield neural networks (CVNNs), we propose two complex valued neural networks based on magnitude as well as phase quantization. In [5], such CVNNs were first proposed by the author. They are described below:

(I) We employ the following complex ceiling function as the activation function. Let u = a + j b be an arbitrary complex number and let S, R be positive phase, magnitude resolution factors respectively.

CoCeil(u) = Ceil (a) + j Ceil (b), where

$$\text{Ceil(a)} \triangleq \begin{cases} 0 & if\ \ 0 \le \text{a} < R \\ 1 & if\ \ R \le \text{a} < 2R \\ \vdots \\ (Q-1) & if\ \ (Q-2)R \le \text{a} < (Q-1)R \\ Q & if\ (Q-1)\,R\ \le\ a\ \le QR \end{cases}.$$

Similarly, we define Ceil(b).

In the serial/asynchronous mode of operation of such complex valued artificial neural network, the state of ith neuron at time 'k' i.e. v_i(k) is updated as follows:

$$v_i(k+1) = CoCeil\left[\textstyle\sum_{j=1}^{L} k_{ij} v_j(k) - t_i\right], where$$

t_i is the threshold at the ith neuron.

In the research papers [7], [8], it is reasoned that the real valued Hopfield neural network based on ceiling neuron satisfies an associated convergence Theorem (exhibiting stability). As in the case of CVNN based on COSIGN activation function [4], the proof argument can be generalized to the CVNN (I) proposed above. In fact LEMMA 1, can also be used to reduce the proof associated with CVNN (I) to the real valued CEILING neuron based Hopfield neural network proposed in [8]. Details are avoided for brevity.

We now propose another CVNN with associated magnitude and phase quantization. It is based on the following COMPLEX CEILING function as the activation function:

Let 'u' be an arbitrary complex number and let S, R be positive phase and magnitude resolution factors respectively.

COCEIL(u) = CEIL(r) $e^{j\ CEIL(\delta)}$, where

$$\text{Ceil(r)} \triangleq \begin{cases} 0 & if\ \ 0 \le \mathrm{r} < R \\ 1 & if\ \ R \le \mathrm{r} < 2R \\ \vdots & \\ (Q-1) & if\ \ (Q-2)R \le \mathrm{r} < (Q-1)R \\ Q & if\ (Q-1)\,R\ \le\ r\ \le QR \end{cases}$$

Also, as in Section 3, phase quantization is performed using the Ceiling activation function.

Proofs of Convergence Theorems:

As in the case of CVNN based on Complex Signum function proposed in Section 3, using the convergence theorem associated with Ceiling neuron, we reason that the energy function (Hermitian form) is non-decreasing with magnitude quantization followed by phase quantization. Thus, in the serial mode the CVNN converges to a stable state. Alternatively,using Lemma 1, it is proved that the quadratic form associated with a Hermitian matrix can be reduced to that associated with two real valued symmetric matrices. Thus, we use the convergence theorem associated with real valued neural networks based on CEILING neuron to prove the convergence to stable state starting with any initial condition.

5. Conclusion:

In this research paper, a novel proof of convergence theorem associated with a CVNN based on magnitude quantization using complex signum function is proved. Also, two novel CVNNs based on magnitude and phase quantization are proposed. The associated convergence theorems are proved.

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