On Maximum-Sum Matchings of Points

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──── **Abstract** ────

Huemer et al. (Discrete Mathematics, 2019) proved that for any two point sets R and B with $|R| = |B|$, the perfect matching that matches points of R with points of B, and maximizes the total *squared* Euclidean distance of the matched pairs, verifies that all the disks induced by the matching have a common point. Each pair of matched points $p \in R$ and $q \in B$ induces the disk of smallest diameter that covers p and q. Following this research line, in this paper we consider the perfect matching that maximizes the total Euclidean distance. First, we prove that this new matching for R and B does not always ensure the common intersection property of the disks. Second, we extend the study of this new matching for sets of $2n$ uncolored points in the plane, where a matching is just a partition of the points into n pairs. As the main result, we prove that in this case all disks of the matching do have a common point. This implies a big improvement on a conjecture of Andy Fingerhut in 1995, about a maximum matching of $2n$ points in the plane.

1 Introduction

Let R and B be two disjoint point sets in the plane with $|R| = |B| = n$, $n \geq 2$. The points in R are *red*, and those in B are *blue*. A *matching* of $R \cup B$ is a partition of $R \cup B$ into n pairs such that each pair consists of a red and a blue point. A point $p \in R$ and a point $q \in B$ are *matched* if and only if the (unordered) pair (p, q) is in the matching. For every $p, q \in \mathbb{R}^2$, we use pq to denote the segment connecting p and q, and $\|p - q\|$ to denote its length, which is the Euclidean norm of the vector $p - q$. Let D_{pq} denote the disk with diameter equal to $\|p - q\|$, that is centered at the midpoint $\frac{p+q}{2}$ of the segment pq. For any matching $\mathcal{M}$, we use $D_{\mathcal{M}}$ to denote the set of the disks associated with the matching, that is, $D_{\mathcal{M}} = \{D_{pq} \mid (p, q) \in \mathcal{M}\}$.

Huemer et al. [14] proved that if $\mathcal{M}$ is any matching that maximizes the total *squared* Euclidean distance of the matched points, i.e., it maximizes $\sum_{(p,q) \in \mathcal{M}} \|p - q\|^2$, then all disks of $D_{\mathcal{M}}$ have a point in common.

In this paper, we will consider the *max-sum* matching $\mathcal{E}$, as the matching that maximizes the total Euclidean distance of the matched points. For any matching $\mathcal{M}$, let $\mathrm{cost}(\mathcal{M})$ denote the sum $\sum_{(p,q) \in \mathcal{M}} \|p - q\|$. Thus, $\mathcal{E}$ is such that $\mathrm{cost}(\mathcal{E})$ is maximum among all matchings. In Section 2, we prove that every pair of disks in $\mathcal{E}$ have a common point, but it cannot be guaranteed that *all disks* have a common point.

In this paper, we will also consider max-sum matchings of sets of $2n$ uncolored points in the plane, where a matching is just a partition of the points into n pairs. Then, we study the following problem: Fingerhut [12] conjectured that given a set P of $2n$ uncolored points in the plane and a max-sum matching $\{(a_i, b_i), i = 1, \ldots, n\}$ of P, there exists a point o of the plane, not necessarily a point of P, such that

$$\|a_i - o\| + \|b_i - o\| \ \leq \ \frac{2}{\sqrt{3}} \cdot \|a_i - b_i\| \quad \text{for all } i \in \{1, \ldots, n\}, \text{ where } 2/\sqrt{3} \approx 1.1547. \quad (1)$$

The statement of equation (1) is equivalent to stating that the intersection $E_1 \cap E_2 \cap \cdots \cap E_n$ is not empty, where E_i is the region bounded by the ellipse with foci a_i and b_i, and semimajor axis length $(1/\sqrt{3}) \cdot \|a_i - b_i\|$, for all $i \in \{1, \ldots, n\}$ [12]. Then, by Helly's Theorem, it is sufficient to prove equation (1) for $n = 3$. As noted by Andy Fingerhut, the factor $2/\sqrt{3}$ is the minimum possible. It is enough to consider an equilateral triangle, where at each vertex two points are located. The max-sum matching of the six points is made of pairs of vertex-opposed points, and the regions defined by the three ellipses using the factor $2/\sqrt{3}$ have exactly one point in common.

Andy Fingerhut was also interested in a small constant close to $2/\sqrt{3}$, and Eppstein [12] proved that the result holds with 2.5 instead of $2/\sqrt{3}$. The proof is simple and does not require Helly's theorem. Let o be the midpoint of the shortest edge in the matching. Namely, for all $i \in \{1, \ldots, n\}$,

$$\|a_i - o\| + \|b_i - o\| \ \leq \ 2.5 \cdot \|a_i - b_i\|. \quad (2)$$

In Section 3, we first show that the constant in this inequality can be improved to $\sqrt{5} \approx 2.236$ by using the same point o, but refined arguments. Second, as the main result of this paper, we improve even further this constant. Precisely, we prove that for any point set P of $2n$ uncolored points in the plane and a max-sum matching $\mathcal{E} = \{(a_i, b_i), i = 1, \ldots, n\}$ of P, all disks in $D_{\mathcal{E}}$ have a common intersection. This directly implies that any point o in the common intersection satisfies

$$\|a_i - o\| + \|b_i - o\| \ \leq \ \sqrt{2} \cdot \|a_i - b_i\|$$

for all $i \in \{1, \ldots, n\}$, where $\sqrt{2} \approx 1.4142$. We will use Helly's theorem, that is, we will prove the claim for $n = 3$.

Remark. Recently, Adiprasito *et al.* [3] proved a no-dimension version of Tverberg's theorem: For any set P of n points in $\mathbb{R}^d$ and $k \in \{2, \ldots, n\}$, there exists a point $o \in \mathbb{R}^d$ and a partition of P into k sets $P_1, \ldots, P_k$ such that, for all $i \in \{1, \ldots, k\}$, $d(o, conv(P_i)) \leq (2 + \sqrt{2})\sqrt{\frac{k}{n}}\, diam(P)$, where $conv(P_i)$ is the convex hull of P_i, $d(o, conv(P_i))$ is the Euclidean distance from o to $conv(P_i)$, and $diam(P)$ is the diameter of P. Our result can improve the bound to $d(o, conv(P_i)) \leq \frac{1}{2}\, diam(P_i) \leq \frac{1}{2}\, diam(P)$ for $d = 2$ and $n = 2k$ (using the max-sum matching for P and taking o in the intersection of the disks). For $d = 2$ and arbitrary values of k, another bound of $d(o, conv(P_i)) \leq \frac{1}{\sqrt{3}}\, diam(P)$ is implied by Jung's theorem [15] (taking as o the center of the smallest disk enclosing P).

1.1 Related problems

As described by Huemer et al. [14], this class of problems is well studied in discrete and computational geometry, starting from the classic result that n red points and n blue points can always be perfectly matched with n pairwise non-crossing segments, where each segment connects a red point with a blue point [16]. The study has been continued in plenty of directions, for both the monochromatic and bichromatic versions, by using pairwise disjoint objects inducing the matching: segments [4, 11], rectangles [1, 2, 6, 7, 9], and more general geometric objects [5].

Our results, as that of [14], are in the direction opposite to that of above mentioned known results on matching points: The goal is that all matching objects (i.e., the disks in $D_{\mathcal{M}}$) have a common intersection, whereas in previous work it is required that all matching objects are pairwise disjoint.

The problem of stabbing a finite set of pairwise intersecting disks with four or five points [8, 10, 13], is also related with this paper, since we deal with finite sets of pairwise intersecting disks.

Particularly, the conjecture posted by Fingerhut [12] relates to an abstract version of a problem in designing communication networks. The given $2n$ points represent nodes that should be connected by a network. The cost $cost(\mathcal{E})$ of a max-sum matching $\mathcal{E}$ is a lower bound on the cost of any feasible network, and the point o is a place where one would like to place the center of a "star" network, in which all given points are connected directly to the center of the star.

2 Red-blue matchings

Let R and B be two disjoint point sets defined as above, where $|R| = |B| = n$, $n \geq 2$. In this section, we prove that the perfect matching $\mathcal{E}$ of R and B that maximizes the total Euclidean distance $cost(\mathcal{E})$ does not always ensure the common intersection property of the disks in $D_{\mathcal{E}}$. Nevertheless, although the common intersection is not always possible, all disks must be pairwise intersecting, as proved in the next proposition.

▶ **Proposition 1.** *Every pair of disks in $D_{\mathcal{E}}$ have a non-empty intersection.*

Proof. Let (a, a') and (b, b') be two different pairs of $\mathcal{E}$, with $a, b \in R$ and $a', b' \in B$. Since $\mathcal{E}$ is a maximum matching, we must have $\|a - b'\| + \|a' - b\| \leq \|a - a'\| + \|b - b'\|$. The equality holds, for instance, when a', a, b, and b' are in this order consecutive vertices of a square. Note that, in the contrary case, we will have $cost((\mathcal{E} \setminus \{(a, a'), (b, b')\}) \cup \{(a, b'), (b, a')\}) > cost(\mathcal{E})$.
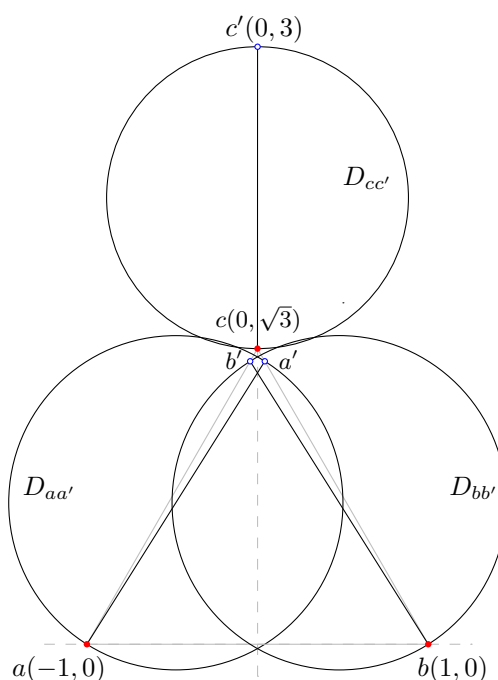
Figure 1 Proof of Theorem 2.

The disks $D_{aa'}, D_{bb'} \in D_{\mathcal{E}}$ have a common point if and only if the distance $\left\| \frac{a+a'}{2} - \frac{b+b'}{2} \right\|$ between the centers $\frac{a+a'}{2}$ and $\frac{b+b'}{2}$ of $D_{aa'}$ and $D_{bb'}$, respectively, is at most the sum $\frac{\|a-a'\|}{2} + \frac{\|b-b'\|}{2}$ of their radii. The following equations ensure this condition:

$$
\begin{aligned}
\|(a + a') - (b + b')\| &= \|(a - b') + (a' - b)\| \\
&\leq \|a - b'\| + \|a' - b\| \\
&\leq \|a - a'\| + \|b - b'\|.
\end{aligned}
$$

Hence, we conclude that $D_{aa'} \cap D_{bb'} \neq \emptyset$ for every pair of disks $D_{aa'}$ and $D_{bb'}$ of $D_{\mathcal{E}}$.   ◄

▶ **Theorem 2.** *There exist point sets $R \cup B$, with $R \cap B = \emptyset$ and $|R| = |B| = 3$, such that, for any max-sum matching $\mathcal{E}$ of R and B, the intersection of the disks of $D_{\mathcal{E}}$ is the empty set.*

Proof. Let $R = \{a, b, c\}$ and $B = \{a', b', c'\}$, with $a = (-1, 0)$, $b = (1, 0)$, $c = (0, \sqrt{3})$, $c' = (0, 3)$, and $a' \in bc$ and $b' \in ac$ such that $\|c - a'\| = \|c - b'\| = \varepsilon$, for a parameter $\varepsilon > 0$ that ensures that $\mathcal{E} = \{(a, a'), (b, b'), (c, c')\}$ is the only maximum matching of $R \cup B$ (see Figure 1).

Note that

$$
\begin{aligned}
\|a - b'\| + \|b - c'\| + \|c - a'\| &= \|a - c'\| + \|b - a'\| + \|c - b'\| \\
&= \sqrt{10} + (2 - \varepsilon) + \varepsilon \\
&= 2 + \sqrt{10},
\end{aligned}
$$

and we need to ensure that

$$
2 + \sqrt{10} < \|a - a'\| + \|b - b'\| + \|c - c'\| = \mathrm{cost}(\mathcal{E}).
$$

That is, the matching $\{(a, a'), (b, b'), (c, c')\}$ has larger total Euclidean distance than the matchings $\{(a, b'), (b, c'), (c, a')\}$ and $\{(a, c'), (b, a'), (c, b')\}$. Since

$$
\begin{aligned}
\|a - a'\| + \|b - b'\| + \|c - c'\| &= 2\|a - a'\| + \|c - c'\| \\
&= 2\|a - a'\| + (3 - \sqrt{3}) \\
&> 2(\|a - c\| - \varepsilon) + (3 - \sqrt{3}) \\
&= 7 - \sqrt{3} - 2\varepsilon,
\end{aligned}
$$

it suffices to ensure

$$
2 + \sqrt{10} \; < \; 7 - \sqrt{3} - 2\varepsilon \quad \Longleftrightarrow \quad \varepsilon \; < \; \frac{5 - \sqrt{10} - \sqrt{3}}{2} \; \approx \; 0.0528. \tag{3}
$$

Furthermore, since

$$
\begin{aligned}
\|a - c'\| + \|b - b'\| + \|c - a'\| &= \|a - a'\| + \|b - c'\| + \|c - b'\| \\
&< (2 + \varepsilon) + \sqrt{10} + \varepsilon \\
&= 2 + \sqrt{10} + 2\varepsilon,
\end{aligned}
$$

to ensure that $\{(a, a'), (b, b'), (c, c')\}$ has larger total Euclidean distance than $\{(a, c'), (b, b'), (c, a')\}$ and $\{(a, a'), (b, c'), (c, b')\}$, it suffices to guarantee that

$$
2 + \sqrt{10} + 2\varepsilon \; < \; 7 - \sqrt{3} - 2\varepsilon \quad \Longleftrightarrow \quad \varepsilon \; < \; \frac{5 - \sqrt{10} - \sqrt{3}}{4} \; \approx \; 0.0264. \tag{4}
$$

Hence, any $\varepsilon > 0$ satisfying (4) (and then also (3)) is such that $\mathcal{E} = \{(a, a'), (b, b'), (c, c')\}$ is the only maximum matching of $R \cup B$. It remains to show that $D_{aa'} \cap D_{bb'} \cap D_{cc'} = \emptyset$. To see that, it is straightforward to show (based on the fact that $c \notin D_{aa'}$ and $c \notin D_{bb'}$) that all points of $D_{aa'} \cap D_{cc'}$ have negative x-coordinates, and all points of $D_{bb'} \cap D_{cc'}$ have positive x-coordinates. ◄

Let $|R| = |B| = n$. If $n = 2$, then the disks of $D_{\mathcal{E}}$ always intersect by Proposition 1. If $n = 3$, then the disks of $D_{\mathcal{E}}$ may not intersect by Theorem 2. One could expect that the disks of $D_{\mathcal{E}}$ always intersect if n is sufficiently large. Next, we answer this question negatively.

▶ **Theorem 3.** *For any $n \geq 4$, there exist point sets $R \cup B$, with $R \cap B = \emptyset$ and $|R| = |B| = n$, such that, for any max-sum matching $\mathcal{E}$ of R and B, the intersection of the disks of $D_{\mathcal{E}}$ is the empty set.*

Proof. We construct a set R of n red points, and a set B of n blue points, as follows. First, take six points $a, b, c \in R$ and $a', b', c' \in B$ as in Theorem 2. Second, we add $n - 3$ red points and $n - 3$ blue points in the ε-neighborhood of point c, as explained below (see Figure 2, a zoomed-in view of Figure 1), where $\varepsilon > 0$ is a sufficiently small number that will be specified later.

Add $n - 3$ blue points, denoted $a_1', a_2', \ldots, a_{n-3}'$, on the segment $b'a'$. Note that they are within distance ε from c since $\|c - b'\| = \|c - a'\| = \varepsilon$. Add $n - 3$ red points, denoted $a_1, a_2, \ldots, a_{n-3}$, on the horizontal line through c (perpendicular to cc'), and within distance ε from c (see Figure 2). Then, we have $\|c' - a_i\| \geq \|c' - c\|$ for all $i \in \{1, \ldots, n - 3\}$.

Consider two matchings $\mathcal{M}_1$ and $\mathcal{M}_2$ such that: in $\mathcal{M}_1$ point c' is matched to c or to some a_i, $i \in \{1, \ldots, n - 3\}$; and in $\mathcal{M}_2$ point c' is matched to a or to b. Given $p \in R \cup B$, and a matching $\mathcal{M}$ of $R \cup B$, let $\mathcal{M}(p)$ be the point such that p and $\mathcal{M}(p)$ are matched in $\mathcal{M}$. Clearly, it holds that

$$
\|c' - \mathcal{M}_1(c')\| \; \geq \; \|c' - c\| \; = \; 3 - \sqrt{3} \;\; \text{and} \;\; \|a - \mathcal{M}_1(a)\| \; \geq \; \|a - c\| - \varepsilon \; = \; 2 - \varepsilon.
$$
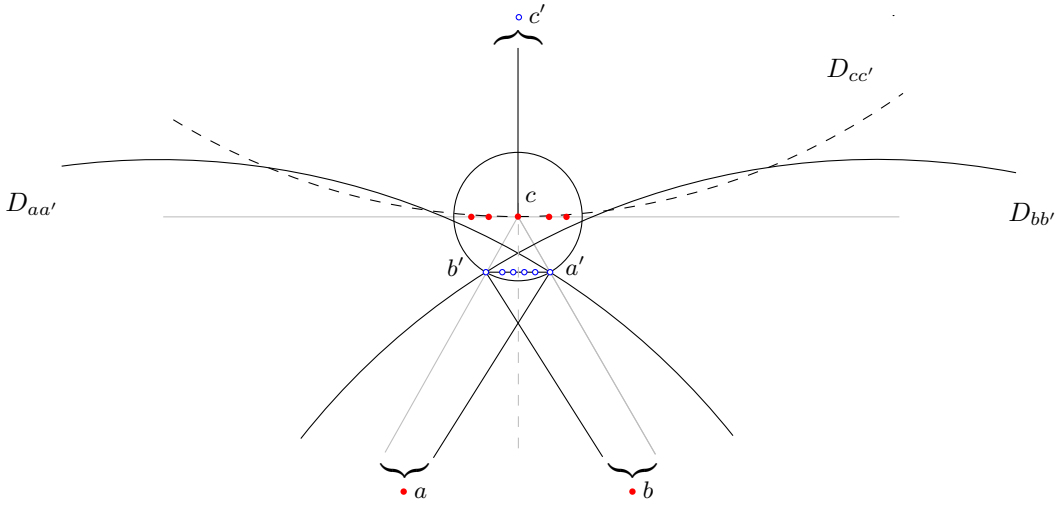
Figure 2 Proof of Theorem 3.

Similarly, $\|b - \mathcal{M}_1(b)\| \geq 2 - \varepsilon$. Then, $\text{cost}(\mathcal{M}_1) \geq 3 - \sqrt{3} + 2(2 - \varepsilon) = 7 - \sqrt{3} - 2\varepsilon$.

By symmetry, we can assume that $(a, c') \in \mathcal{M}_2$. In $\mathcal{M}_2$, point b is matched to a', to b', or to some a_i', $i \in \{1, \ldots, n-3\}$. Then, we have that

$$\|b - \mathcal{M}_2(b)\| \ \leq \ \|b - b'\| \ \leq \ \|b - c\| + \varepsilon \ = \ 2 + \varepsilon.$$

Furthermore, the members of the remaining $n - 2$ pairs of $\mathcal{M}_2$ are within distance ε from c. Thus, $\text{cost}(\mathcal{M}_2) \leq \sqrt{10} + 2 + \varepsilon + 2(n-2)\varepsilon$. We choose $\varepsilon > 0$ such that

$$7 - \sqrt{3} - 2\varepsilon \ > \ \sqrt{10} + 2 + \varepsilon + 2(n-2)\varepsilon,$$

or what is the same,

$$5 - \sqrt{3} - \sqrt{10} \ > \ (2n - 1)\varepsilon.$$

Since $5 - \sqrt{3} - \sqrt{10} \approx 0.10567$, we can choose $\varepsilon < \frac{1}{10(2n-1)}$ to ensure $\text{cost}(\mathcal{M}_1) > \text{cost}(\mathcal{M}_2)$.

Therefore, $\mathcal{E} \neq \mathcal{M}_2$ and we can assume that $\mathcal{E} = \mathcal{M}_1$. To show that the intersection of the disks in $D_{\mathcal{E}}$ is empty, it suffices to show that the disks $D_{a\mathcal{M}_1(a)}, D_{b\mathcal{M}_1(b)}$ and $D_{c'\mathcal{M}_1(c')}$ do not have a common point. To this end, note that (which is straightforward to prove by construction) that all points of $D_{a\mathcal{M}_1(a)} \cap D_{c'\mathcal{M}_1(c')}$ have negative x-coordinates, and all points of $D_{b\mathcal{M}_1(b)} \cap D_{c'\mathcal{M}_1(c')}$ have positive x-coordinates. The result thus follows. ◀

3   On Fingerhut's Conjecture

In this section, we first show that the constant in inequality (2) can be improved to $\sqrt{5} \approx 2.236$. We start by proving the next technical lemma:

▶ **Lemma 4.** *Let p and q be two points of the plane, and let r_{pq} be the radius of disk D_{pq}. Let D be a second disk with center o and radius $r \leq r_{pq}$ such that $D \cap D_{pq} \neq \emptyset$. Then,*

$$\|p - o\| + \|q - o\| \ \leq \ \sqrt{5} \cdot \|p - q\|.$$

Proof. Let o_{pq} denote the center of D_{pq}, and note that $\|o - o_{pq}\| \le r + r_{pq} \le 2 \cdot r_{pq} = \|p - q\|$, because $D \cap D_{pq} \ne \emptyset$. Then, we have that:

$$
\begin{aligned}
(\|p - o\| + \|q - o\|)^2 &\le 2 \cdot (\|p - o\|^2 + \|q - o\|^2) && \text{(Cauchy-Schwarz's inequality)} \\
&= 2 \left(\frac{1}{2} \|p - q\|^2 + 2 \cdot \|o - o_{pq}\|^2 \right) && \text{(Apollonius' theorem)} \\
&= \|p - q\|^2 + 4 \cdot \|o - o_{pq}\|^2 \\
&\le 5 \cdot \|p - q\|^2,
\end{aligned}
$$

which implies the lemma. ◄

Lemma 4 and Proposition 1 imply the following theorem.

▶ **Theorem 5.** *For any $n \ge 2$, and any planar n-point sets R and B such that $R \cap B = \emptyset$, consider any max-sum matching $\mathcal{E} = \{(a_i, b_i), i = 1, \dots, n\}$ of $R \cup B$. Let o be the midpoint of the shortest segment in the matching $\mathcal{E}$. Then, for all $i \in \{1, \dots, n\}$,*

$$
\|a_i - o\| + \|b_i - o\| \le \sqrt{5} \cdot \|a_i - b_i\|.
$$

Note that the bound of $\sqrt{5}$ is tight if the point o is always the midpoint of the shortest segment. Namely, consider 2 red points and 2 blue points, as vertices of a square, such that diagonal-opposed vertices have the same color.

We now strengthen Theorem 5. Precisely, we prove that for any point set P of $2n$ uncolored points in the plane and a max-sum matching $\mathcal{M} = \{(a_i, b_i), i = 1, \dots, n\}$ of P, all disks in $D_{\mathcal{M}}$ have a common intersection. We will use Helly's theorem, that is, we will prove the claim for $n = 3$.

We note that P might contain different points with the same coordinates. Furthermore, the common intersection might be a singleton. Namely, consider six points a, b, c, a', b', and c', where a, b, and c are the vertices of a non-empty triangle, and a', b', and c' coincide with a point z in the interior of the triangle. By the triangle inequality, the matching $\{(a, a'), (b, b'), (c, c')\}$ is max-sum, and z is the only point in the common intersection $D_{aa'} \cap D_{bb'} \cap D_{cc'}$. For any matching $\mathcal{M}$, the *segments* of $\mathcal{M}$ is the segment set $\{pq \mid (p, q) \in \mathcal{M}\}$.

Let p, q, and r be three points of the plane. We denote by Δpqr the triangle with vertices p, q, and r; by $\ell(p, q)$ the straight line through p and q oriented from p to q; by $\tau(p, q)$ the ray with apex p that goes through q; by $\vec{pq}$ the segment pq oriented from p to q; and by C_{pq} the circle bounding D_{pq}. Furthermore, given a fourth point s, we say that $\vec{pq}$ *points to* rs if q is in the interior of $\Delta prs \cap D_{rs}$. See Figure 3 (left), where segment $\vec{cd}$ points to ab.

The first observation that we state is that if $\mathcal{M}$ is a max-sum matching, then the disks of $D_{\mathcal{M}}$ intersect pairwise, as a consequence of Proposition 1. Second, every pair of segments of $\mathcal{M}$ either cross or one oriented segment points to the other one, as formally proved in the following lemma. Third, the four vertices of any two segments cannot be in convex position, because the matching would not be max-sum by the triangle inequality.

▶ **Lemma 6.** *Let $\{a, b, c, d\}$ be a set of four points such that $\{(a, b), (c, d)\}$ is a max-sum matching of $\{a, b, c, d\}$ and d belongs to the interior of Δabc. Then, d belongs to the interior of disk D_{ab}. That is, $\vec{cd}$ points to ab.*

Proof. Let d' be the reflection of d about the midpoint $m = (a + b)/2$ of segment ab, also the center of D_{ab}, and assume w.l.o.g. that d belongs to triangle $\Delta acd'$. Refer to Figure 3. Then,

$$
\|c - d\| + \|d - d'\| < \|c - a\| + \|a - d'\|.
$$
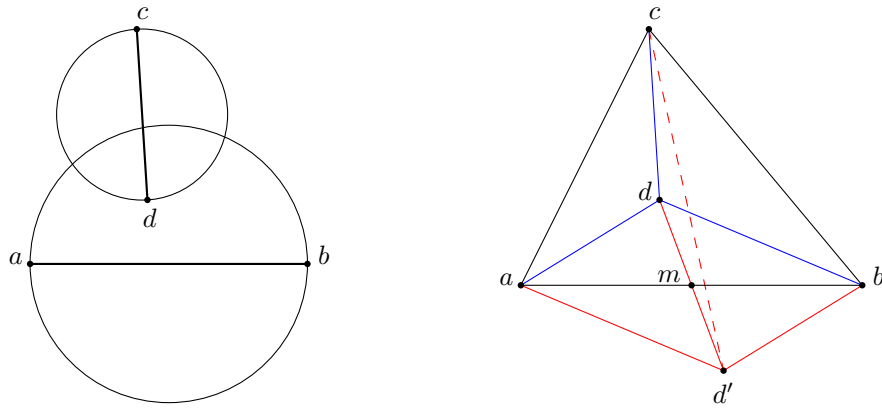
Namely, the perimeter of triangle $\Delta dcd'$ is smaller than the perimeter of triangle $\Delta acd'$, which implies the equation. Using that $\|a - d'\| = \|d - b\|$ because the quadrilateral with vertex set $\{a, d', b, d\}$ is a parallelogram (or rhomboid), and that $\|c - a\| + \|d - b\| \leq \|a - b\| + \|c - d\|$ because $\{(a, b), (c, d)\}$ is a max-sum matching, the equation can be extended to

$$
\begin{aligned}
\|c - d\| + \|d - d'\| \ &< \ \|c - a\| + \|a - d'\| \\
&= \ \|c - a\| + \|d - b\| \\
&\leq \ \|a - b\| + \|c - d\|,
\end{aligned}
$$

which implies that $\|d - d'\| < \|a - b\|$. That is, $\|d - m\| = \|d - d'\|/2$ is smaller than the radius $\|a - m\| = \|a - b\|/2$ of D_{ab}, whose center is m. These observations imply the result.   ◄

As mentioned above, we will elaborate a proof for $n = 3$, that is, matchings of three segments. Since every pair of segments of a max-sum matching cross, or one of them points to the other one, we distinguish ten cases of relative position (or order-type) of the three segments, as shown in Figure 4, enumerated from **(A)** to **(J)**. In the rest of this section, we will first show with direct proofs that in each of the cases from **(A)** to **(G)** the three disks have a common point. After that, we will use a proof by contradiction for each of the cases from **(H)** to **(J)**.

The general idea of these latter proofs by contradiction is the following: Since the disks pairwise intersect, but they do not have a common intersection, we can extend one of the segments (by moving one of its vertices) such that the new three disks have a singleton common intersection. In this new setting, the three new segments must fall again in one of the cases from **(H)** to **(J)**. Then, we show that the new segments must correspond to a max-sum matching given that the original ones do, but contradictorily such a new matching is not max-sum.

▶ **Lemma 7.** *If the segments of a max-sum matching of six points fall in one of the cases from* **(A)** *to* **(G)***, then the three disks of the matching have a common intersection.*

Proof. Let $\{a, b, c, a', b', c'\}$ be a 6-point set, and let $\mathcal{M} = \{(a, a'), (b, b'), (c, c')\}$ be a max-sum matching. In any case, refer to Figure 5 for the location of each point.

Case **(A)**: At least one altitude of the triangle T bounded by the three segments goes through the interior of T. Let u be the vertex of such an altitude in a side of T. By Thales' theorem, each of the three disks $D_{aa'}$, $D_{bb'}$, and $D_{cc'}$ contains u.
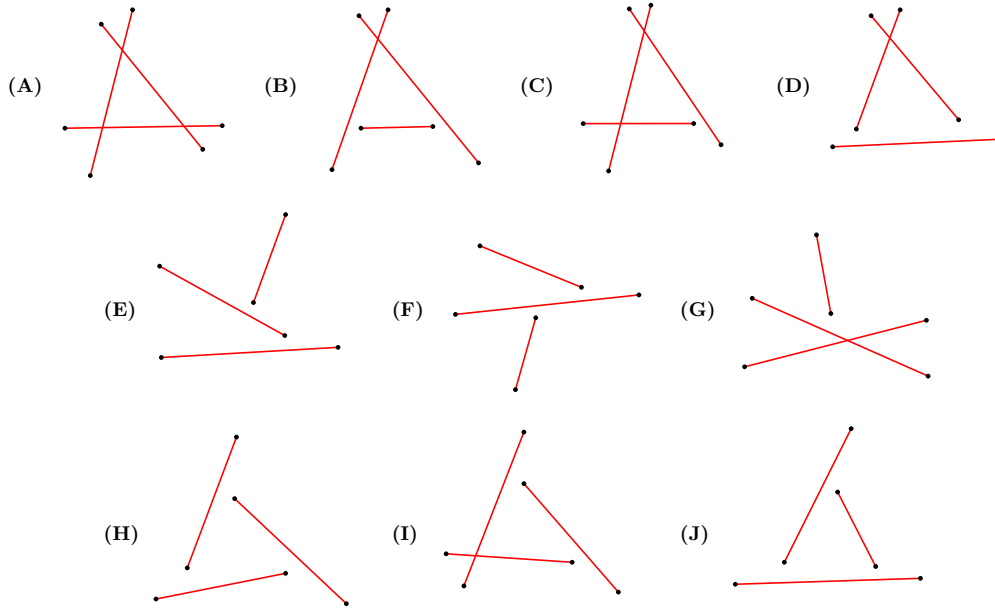
Figure 4 The ten different relative positions of three segments.

Case **(B)**: Let u be the intersection point between bb' and cc'. If $D_{bb'}$ contains a, then we are done since $a \in D_{cc'}$ because $\vec{a'a}$ points to cc' (Lemma 6). Similarly, if $D_{cc'}$ contains a', then we are done since $a' \in D_{bb'}$ because $\vec{aa'}$ points to bb'. Otherwise, if $a \notin D_{bb'}$ and $a' \notin D_{cc'}$, then the triangle $\Delta aa'u$ is such that the interior angles at a and a', respectively, are both acute. Hence, the altitude h from vertex u goes through the interior of $\Delta aa'u$, and let $v \in aa'$ be the other vertex of h. Since $\vec{a'a}$ points to cc', and $\vec{aa'}$ points to bb', each of the disks $D_{aa'}$, $D_{bb'}$, and $D_{cc'}$ contains v, by Thales' theorem. The proof for Case **(C)** is both similar and simpler.

Case **(D)**: If $D_{bb'}$ contains c', then we are done since $c' \in D_{aa'}$ because $\vec{cc'}$ points to aa' (Lemma 6). Similarly, if $D_{cc'}$ contains b, then we are done since $b \in D_{aa'}$ because $\vec{b'b}$ points to aa'. Otherwise, if $c' \notin D_{bb'}$ and $b \notin D_{cc'}$, then the triangle $\Delta c'bu$ is such that the interior angles at c' and b, respectively, are both acute. Hence, the altitude h from vertex u goes through the interior of $\Delta c'bu$, and let $v \in c'b$ be the other vertex of h. We have $v \in D_{bb'} \cap D_{cc'}$, by Thales' theorem. Furthermore, since $c', b \in D_{aa'}$, we have that segment $c'b$ is contained in $D_{aa'}$. Hence, $v \in D_{aa'} \cap D_{bb'} \cap D_{cc'}$.

Cases **(E)**, **(F)**, and **(G)**: In each of these cases, the same oriented segment points to each of the other two ones: Say, segment $\vec{aa'}$ points to both bb' and cc'. Then, we have that $a' \in D_{bb'} \cap D_{cc'}$, by Lemma 6. Hence, $a' \in D_{aa'} \cap D_{bb'} \cap D_{cc'}$. This completes the proof. ◀

We now prove several technical lemmas, which will be used for the cases from **(H)** to **(J)**. The following lemma guarantee that if we extend one segment by moving one of the points, then the resulting segments correspond to a max-sum matching of the resulting point set. The rest of the lemmas impose monotone properties, and situations in which $\{(a, a'), (b, b'), (c, c')\}$ is not a max-sum matching of point set $\{a, b, c, a', b', c'\}$, all of them associated with cases from **(H)** to **(J)**.

▶ **Lemma 8.** *Let* $\mathcal{M} = \{(a_i, b_i), i = 1, \ldots, n\}$ *be a max-sum matching of the set P of $2n$ uncolored points, and let $c \notin P$ be a point such that b_1 belongs to the interior of the segment a_1c. Then, $\mathcal{M}^* = (\mathcal{M} \setminus \{(a_1, b_1)\}) \cup \{(a_1, c)\}$ is a max-sum matching of $(P \setminus \{b_1\}) \cup \{c\}$.*
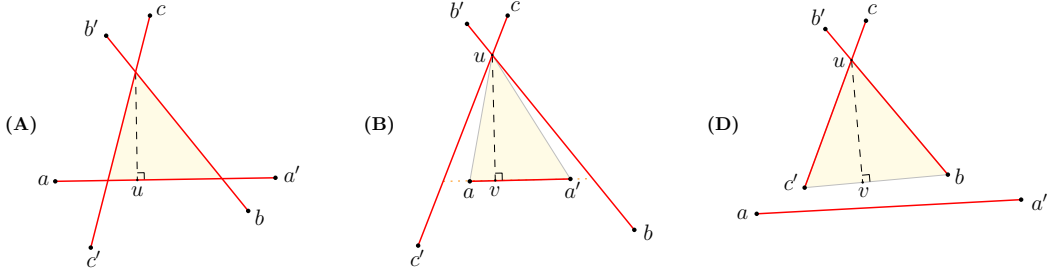
■ **Figure 5** Proof of Lemma 7.

Proof. Let $\mathcal{M}'$ denote any matching of $(P \setminus \{b_1\}) \cup \{c\}$, and note that $(\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}) \cup \{(b_1, \mathcal{M}'(c))\}$ is a matching of P. Then,

$$
\begin{aligned}
\text{cost}(\mathcal{M}') &= \text{cost}\left(\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}\right) + \|\mathcal{M}'(c) - c\| \\
&\leq \text{cost}\left(\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}\right) + \|\mathcal{M}'(c) - b_1\| + \|b_1 - c\| \quad \text{(triangle inequality)} \\
&= \text{cost}((\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}) \cup \{(b_1, \mathcal{M}'(c))\}) + \|b_1 - c\| \\
&\leq \text{cost}(\mathcal{M}) + \|b_1 - c\| \\
&= \text{cost}(\mathcal{M}^*).
\end{aligned}
$$

Hence, the lemma follows. ◀

▶ **Lemma 9.** *Let p, p', q, and q' be four points such that $\vec{pp'}$ points to qq', and q is to the right of $\ell(p, p')$. Let z be a point to the left of both $\ell(p, p')$ and $\ell(q, q')$ such that: (i) q is to the left of $\ell(z, p)$; (ii) vectors $p - z$ and $p' - z$ are orthogonal, and vectors $q - z$ and $q' - z$ are also orthogonal. Refer to Figure 6a. Then, we have that*

$$\|p - z\| - \|q - z\| \;<\; \|p - q'\| - \|q - q'\|. \tag{5}$$

Proof. Rearranging terms in equation (5), we need to prove that

$$\|p - z\| + \|q - q'\| \;<\; \|p - q'\| + \|q - z\|. \tag{6}$$

Conditions (i) and (ii), the fact that $\vec{pp'}$ points to qq', and the location of z, imply that q is to the right of $\ell(z, p')$ if and only if segments pq' and qz have a common point.

Suppose that q is to the right of $\ell(z, p')$ (see Figure 6b), case where segments pq' and qz have a common point. Then, p, q, q', and z are the vertices of a convex quadrilateral with non-empty interior and diagonals pq' and zq. Hence, Equation (6) holds by the triangle inequality.

Suppose now that q is not to the right of $\ell(z, p')$ (see Figure 6c), then segments pq' and qz do not intersect. Let z' be the reflection of z about the center of segment qq', also the center of $D_{qq'}$. Then, we have $\|q - z\| = \|q' - z'\|$ and $\|q - q'\| = \|z - z'\|$, by condition (ii) and Thales' theorem. The fact that p' belongs to the interior of $D_{qq'}$ given that $\vec{pp'}$ points to qq', implies that z' must be to the right of line $\ell(p, z)$. Then, since pq' and qz do not intersect, we have that z belongs to triangle $\Delta pz'q'$. This implies

$$\|p - z\| + \|z - z'\| \;<\; \|p - q'\| + \|q' - z'\|,$$

then equation (5) and the result. ◀
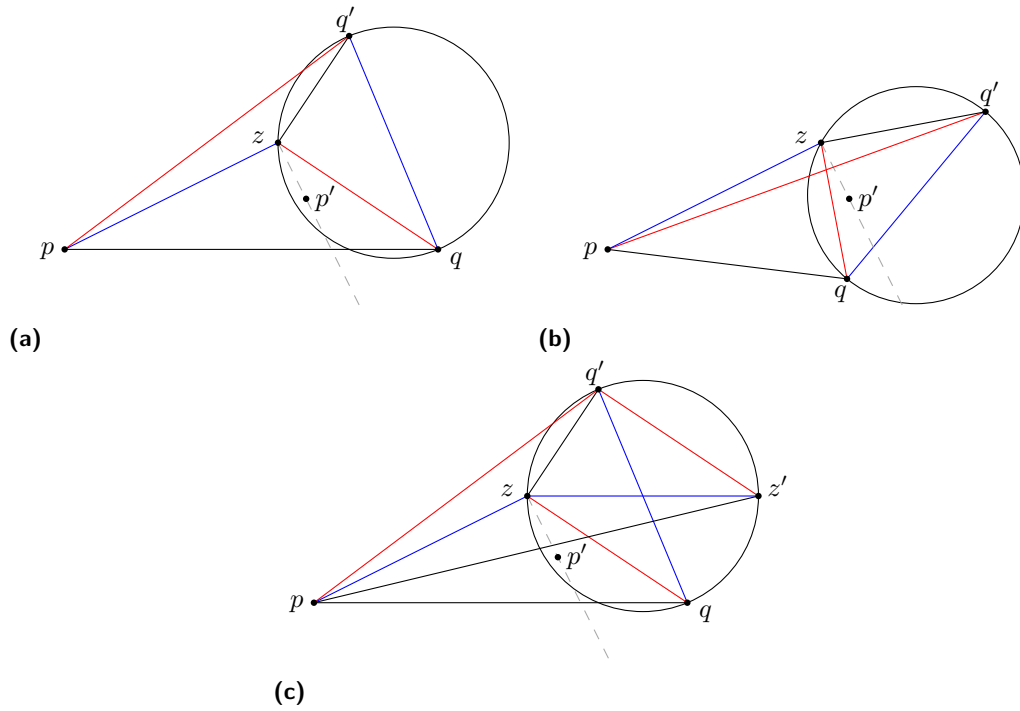
Figure 6 (a) Statement of Lemma 9. (b-c) Proof of Lemma 9.

▶ **Lemma 10.** *Let p, p', q, and q' be four points in convex position such that q and q' are to the right and left of line $\ell(p,p')$, respectively. Let z be a point to the left of both $\ell(p,p')$ and $\ell(q,q')$ such that: vectors $p-z$ and $p'-z$ are orthogonal, and vectors $q-z$ and $q'-z$ are also orthogonal. Then, we again have that $\|p-z\| - \|q-z\| < \|p-q'\| - \|q-q'\|$.*

Proof. The proof is now simpler. In this case, segments pq' and qz have a common point, and the proof continues as that of Lemma 9. ◀

▶ **Lemma 11.** *Let a, b, c, a', b', c', and z be seven points such that: c is to the left of line $\ell(a,b)$; segments $a\vec{a'}$, $b\vec{b'}$, and $c\vec{c'}$ point to bb', cc', and aa', respectively; and for each $u \in \{a,b,c\}$, point z is to the left of line $\ell(u,u')$, and vectors $u-z$ and $u'-z$ are orthogonal. Refer to Figure 7a. Then, $\{(a,a'),(b,b'),(c,c')\}$ is not a max-sum matching of point set $\{a,b,c,a',b',c'\}$.*

Proof. We will prove

$$\|a-a'\| + \|b-b'\| + \|c-c'\| \;<\; \|a-b'\| + \|b-c'\| + \|c-a'\|. \tag{7}$$

The conditions of the lemma guarantee (three times) the conditions of Lemma 9. That is, we can apply Lemma 9 for a, a', b, b', and z (where a and b play the role of p and q, respectively); for b, b', c, c', and z (where b and c play the role of p and q, respectively); and for c, c', a, a', and z (where c and a play the role of p and q, respectively). Hence, we obtain the following three inequalities:

$$\|a-z\| - \|b-z\| \;<\; \|a-b'\| - \|b-b'\|, \qquad \text{(Lemma 9)}$$
$$\|b-z\| - \|c-z\| \;<\; \|b-c'\| - \|c-c'\|, \qquad \text{(Lemma 9)}$$
$$\|c-z\| - \|a-z\| \;<\; \|c-a'\| - \|a-a'\|. \qquad \text{(Lemma 9)}$$
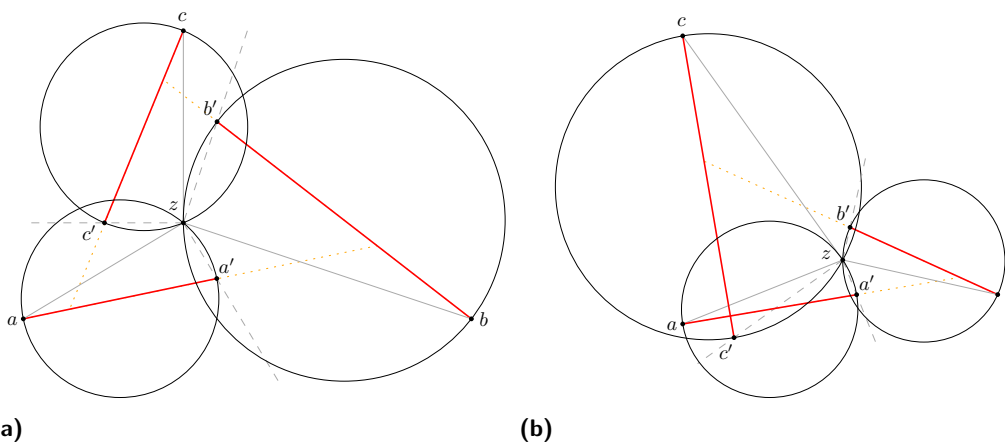
Figure 7 (a) Lemma 11. (b) Lemma 12.

Adding the above three inequalities, we obtain inequality (7). The result thus follows. ◀

▶ **Lemma 12.** *Let a, b, c, a', b', c', and z be seven points such that: c is to the left of line $\ell(a, b)$; segments $\vec{aa'}$ and $\vec{bb'}$ point to bb' and cc', respectively; segments aa' and cc' have a common point with a and a' to the right and left of line $\ell(c, c')$, respectively; and for each $u \in \{a, b, c\}$, point z is to the left of line $\ell(u, u')$, and vectors $u - z$ and $u' - z$ are orthogonal. Refer to Figure 7b. Then, $\{(a, a'), (b, b'), (c, c')\}$ is not a max-sum matching of point set $\{a, b, c, a', b', c'\}$.*

Proof. The conditions of the lemma guarantee (twice) the conditions of Lemma 9. That is, we can apply Lemma 9 for a, a', b, b', and z (where a and b play the role of p and q, respectively); and for b, b', c, c', and z (where b and c play the role of p and q, respectively). Furthermore, we can apply Lemma 10 for c, c', a, a', and z (where c and a play the role of p and q, respectively). Hence, we obtain the following three inequalities:

$$\|a - z\| - \|b - z\| \;<\; \|a - b'\| - \|b - b'\|, \qquad\qquad \text{(Lemma 9)}$$
$$\|b - z\| - \|c - z\| \;<\; \|b - c'\| - \|c - c'\|, \qquad\qquad \text{(Lemma 9)}$$
$$\|c - z\| - \|a - z\| \;<\; \|c - a'\| - \|a - a'\|. \qquad\qquad \text{(Lemma 10)}$$

Adding the three inequalities above, we obtain again inequality (7). The result thus follows.
◀

Let α be a planar (open or closed) curve that splits the plane into two open regions. Given a point p not in α, let $H(\alpha, p)$ denote the region (between the two above ones) that contains p.

▶ **Proposition 13.** *Let a, b, a', and b' be four points such that $\{(a, a'), (b, b')\}$ is a max-sum matching of $\{a, b, a', b'\}$. Let α be the arc of the hyperbola with foci a and b that goes through b'. Then, we have that $a' \in \alpha \cup H(\alpha, b)$.*

Proof. The arc α is the locus of the points x of the plane such that $\|a - x\| - \|b - x\| = \|a - b'\| - \|b - b'\|$. Since $\{(a, a'), (b, b')\}$ is a max-sum matching, we have that $\|a - b'\| - \|b - b'\| \le \|a - a'\| - \|b - a'\|$, which implies the proposition. ◀
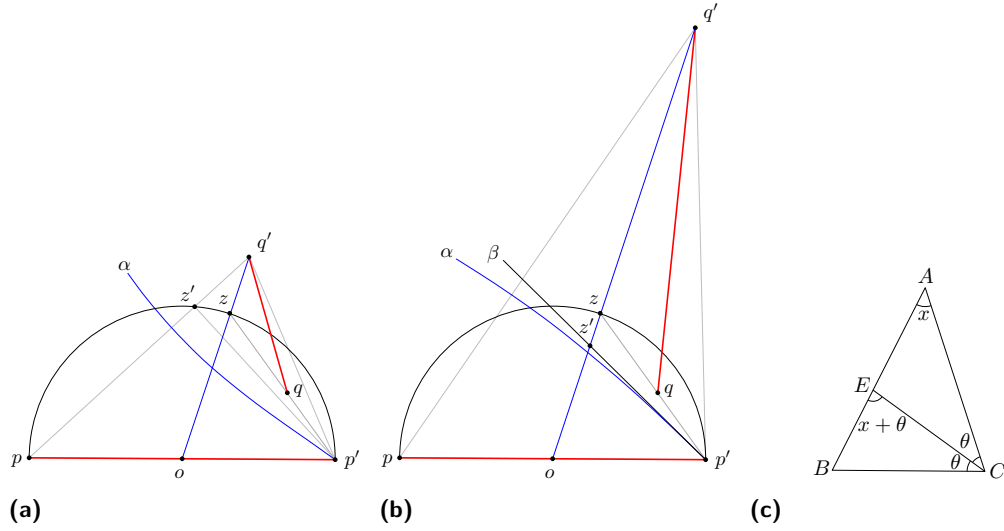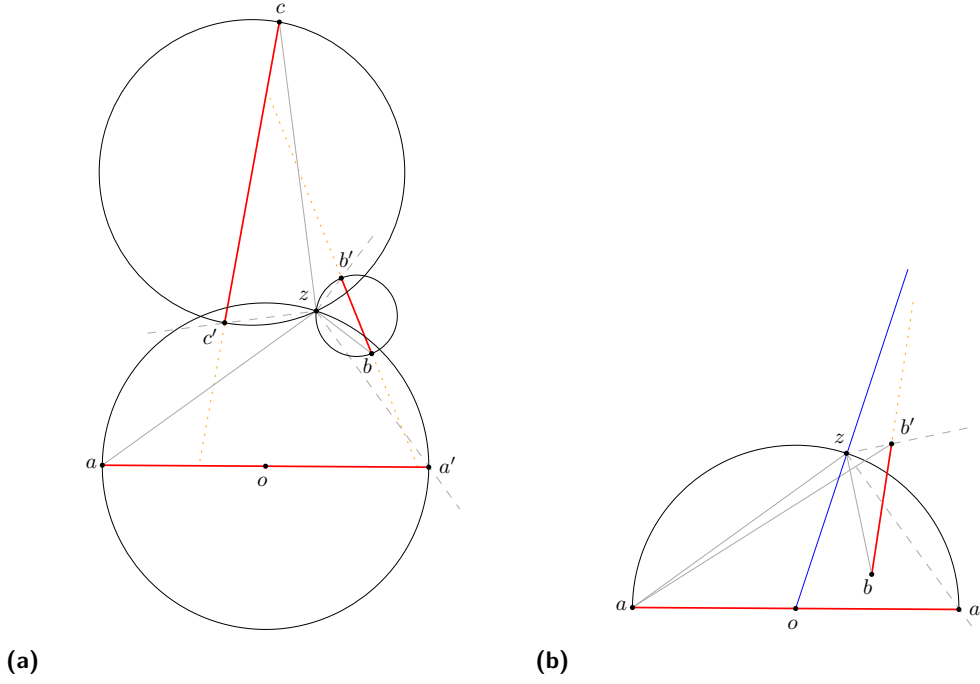
Figure 8 (a,b,c) Proof of Lemma 14.

▶ **Lemma 14.** *Let p, p', and o be three points such that o is the midpoint of segment pp'. Let z be a point of the circle $C_{pp'}$ to the left of line $\ell(p, p')$, q a point of segment zp' with $q \neq p'$, and q' a point of ray $\tau(o, z)$ not in segment oz. Then, $\|p - p'\| + \|q - q'\| < \|p - q\| + \|p' - q'\|$.*

Proof. We divide the proof into two cases: $\|p - p'\| \geq \|q' - p'\|$; and $\|p - p'\| < \|q' - p'\|$. In both cases, let α be the arc of the hyperbola with foci p and q' that goes through p'.

In the first case (see Figure 8a), let z' be the intersection point of $C_{pp'}$ and pq'. Note that $\|p - p'\| \geq \|q' - p'\|$ implies that the region $H(\alpha, q')$ is convex. Furthermore, line $\ell(p', z')$ is perpendicular to the line $\ell(p, q')$ through the foci of α, and this implies that z' belongs to $H(\alpha, q')$, and then z belongs to the convex intersection $H(\ell(p', z'), q') \cap H(\alpha, q')$. Given that z belongs to the interior of $\alpha \cup H(\alpha, q')$, p' is on the boundary of $\alpha \cup H(\alpha, q')$, and that $q \in zp'$, $q \neq p'$, we also have that q belongs to $H(\alpha, q')$. This last fact, equivalent to $q \notin \alpha \cup H(\alpha, p)$, implies $\|q' - q\| - \|p - q\| < \|q' - p'\| - \|p - p'\|$, by Proposition 13, hence the result.

Consider now the second case, $\|p - p'\| < \|q' - p'\|$ (see Figure 8b). Let β be the bisector of the interior angle at p' of triangle $\Delta op'q'$. By well-known properties of hyperbolas, β is the tangent of α at p'. Furthermore, $\|p - p'\| < \|q' - p'\|$ implies that β separates α from vertex q'. Let z' be the intersection point between oq' and β. We use the following claim to show that $\|o - z'\| < \|o - p'\|$: If A, B, and C are the vertices of a triangle, and point E belongs to side AB, such that line $\ell(C, E)$ is the bisector of the interior angle at C, then $\|C - B\| > \|B - E\|$. The claim follows from the fact that in any triangle, precisely in ΔBCE, larger sides correspond to larger opposed interior angles (see Figure 8c). Applying the claim to $\Delta op'q'$, we have $\|o - z'\| < \|o - p'\|$, which implies that β separates point z and arc α. Furthermore, β separates point q and α because $q \in zp' \setminus \{p'\}$. This last fact shows that q is to left of α in the direction from q' to p (i.e. $q \notin \alpha \cup H(\alpha, p)$), implying $\|q' - q\| - \|p - q\| < \|q' - p'\| - \|p - p'\|$, by Proposition 13. The results thus follows. ◀

▶ **Lemma 15.** *Let a, b, c, a', b', c', and z be seven points such that: none of them is to the right of line $\ell(a, a')$; segments $\vec{b'b}$, $\vec{bb'}$, and $\vec{cc'}$ point to aa', cc', and aa', respectively; and for each $u \in \{a, b, c\}$, point z is to the left of line $\ell(u, u')$, and vectors $u - z$ and $u' - z$ are orthogonal. Refer to Figure 9a. Then, $\{(a, a'), (b, b'), (c, c')\}$ is not a max-sum matching of*

(a)   **(b)**

Figure 9 (a,b) Proof of Lemma 15.

$\{a, b, c, a', b', c'\}$.

Proof. We can apply Lemma 9 for b, b', c, c', and z (where b and c play the role of p and q, respectively); and for c, c', a, a', and z (where c and a play the role of p and q, respectively). Hence, we obtain the following two inequalities:

$$\|b - z\| - \|c - z\| \;<\; \|b - c'\| - \|c - c'\|, \qquad\qquad \text{(Lemma 9)}$$
$$\|c - z\| - \|a - z\| \;<\; \|c - a'\| - \|a - a'\|. \qquad\qquad \text{(Lemma 9)}$$

Let o denote the midpoint of segment aa', also the center of $D_{aa'}$. Given that $z - c$ and $z - c'$ are orthogonal, and $\vec{cc'}$ points to aa', we have that c is to the left of line $\ell(o, z)$. Note that if c is to the right of, or in, $\ell(o, z)$, then c' would not belong to the interior of $D_{aa'}$. Similarly, since $z - b$ and $z - b'$ are orthogonal, and $\vec{b'b}$ points to aa', we have that b' is to the right of line $\ell(o, z)$. Then, since $\vec{bb'}$ points to cc', we have that rays $\tau(b, b')$ and $\tau(o, z)$ must intersect.

If point b belongs to triangle $\Delta aa'z$ (see Figure 9b), then the facts that $z - b$ and $z - b'$ are orthogonal, and $\tau(b, b') \cap \tau(o, z) \neq \emptyset$ imply that segments bz and ab' have a common point. Hence, $\|a - z\| + \|b - b'\| < \|b - z\| + \|a - b'\|$ by the triangle inequality. That is,

$$\|a - z\| - \|b - z\| \;<\; \|a - b'\| - \|b - b'\|.$$

Adding the three inequalities above, we obtain again inequality (7), implying the result.

In the contrary case, b does not belong to $\Delta aa'z$, (see Figure 9a), let us assume by contradiction that the matching $\{(a, a'), (b, b'), (c, c')\}$ is max-sum. Then, the matching $\{(a, a'), (b, b')\}$ is also max-sum. Let w and w' be the intersection points of $\ell(b, b')$ with $\tau(o, z)$ and za', respectively. Note that $bb' \subset ww'$, and by Lemma 8, $\{(a, a'), (w, w')\}$ is a max-sum matching of $\{a, a', w, w'\}$. But we can apply Lemma 14, where a and w play the role of p and q, respectively, to have that $\{(a, a'), (w, w')\}$ is not max-sum, which implies the result. ◀
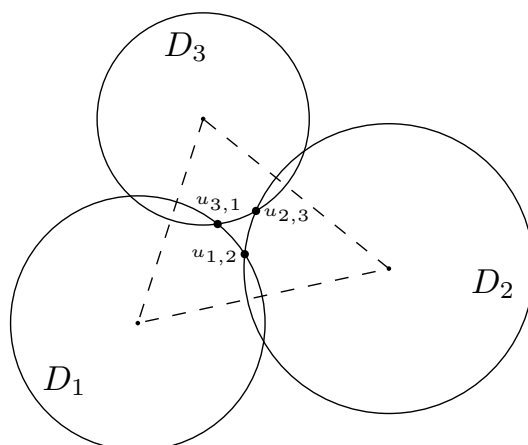
Figure 10 Proof of Lemma 16.

We are now ready to give the proofs to the missing cases, those from **(H)** to **(J)**.

▶ **Lemma 16.** *If the segments of a max-sum matching of six points fall in one of the cases from* **(H)** *to* **(J)**, *then the three disks of the matching have a common intersection.*

Proof. Suppose by contradiction that the three disks, denoted D_1, D_2, and D_3, intersect pairwise, but without a common intersection (see Figure 10). Let $u_{1,2}$, $u_{2,3}$, and $u_{3,1}$ be the vertices of the pairwise disjoint lenses $D_1 \cap D_2$, $D_2 \cap D_3$, and $D_3 \cap D_1$, respectively, located in the triangle with vertices the centers of D_1, D_2, and D_3, respectively.

The idea is to use Lemma 8, in combination with Lemmas 11, 12, and 15, such that the point z of these lemmas is among $u_{1,2}$, $u_{2,3}$, and $u_{3,1}$. To this end, we need to guarantee that point z is not an extreme point of some segment of the matching. This is done in the next paragraph.

Note that two vertices among $u_{1,2}$, $u_{2,3}$, and $u_{3,1}$ cannot be the extreme points of a same segment of the matching. Furthermore, if each of the three vertices is an extreme point of some segment of the matching, then at least one pair of disjoint segments violates Lemma 6. That is, the extreme point of one segment, in the interior of the convex hull of the four involved points, is not in the interior of the disk corresponding to the other segment. Hence, we can assume that at least one vertex among $u_{1,2}$, $u_{2,3}$, and $u_{3,1}$ is not an extreme point of a segment of the matching: say vertex $u_{1,2}$. This implies that we can extend the segment of disk D_3 by moving one of its extreme points such that the new three matching disks have a singleton common intersection at $u_{1,2}$. Let $z = u_{1,2}$, where z is distinct from all the new six points.

Let the new six points be denoted as a, b, c, a', b', and c', in such a way that the new segments are precisely aa', bb', and cc', and for each $u \in \{a, b, c\}$ point z is to the left of line $\ell(u, u')$. By Lemma 8, $\{(a, a'), (b, b'), (c, c')\}$ is a max-sum matching of $\{a, b, c, a', b', c'\}$.

It is important to note the following: If the original segments are in Case **(H)**, then by extending one segment we can stay in Case **(H)**, or go to Case **(I)**. If the original segments are in Case **(I)**, then by extending one segment we can stay in Case **(I)**, or go to Case **(C)** with a non-singleton common intersection of the three disks. Otherwise, if the original segments are in Case **(J)**, then by extending one segment we can stay in Case **(J)**, or go to Case **(B)** or **(D)** with a non-singleton common intersection of the three disks. Hence, since the common intersection of the new three disks $D_{aa'}$, $D_{bb'}$, and $D_{cc'}$ is singleton, we can

ensure that the new segments aa', bb', and cc' are again in a case from **(H)** to **(J)**, and the proof continues as follows.

If aa', bb', and cc' fall in Case **(H)**, then by Lemma 11 $\{(a, a'), (b, b'), (c, c')\}$ is not max-sum. If they are in in Case **(I)**, then by Lemma 12 $\{(a, a'), (b, b'), (c, c')\}$ is not max-sum. Otherwise, if they are in in Case **(J)**, then by Lemma 15 we have that $\{(a, a'), (b, b'), (c, c')\}$ is not max-sum. There exists a contradiction in each of the cases, thus the original three disks must have a common intersection. The lemma thus follows.   ◄

Combining Lemma 7 and Lemma 16, with Helly's theorem, we state now the main results of this paper:

▶ **Theorem 17.** *Let P be a set of $2n$ (uncolored) points in the plane. Any max-sum matching $\mathcal{M}$ of P is such that all disks of $D_{\mathcal{M}}$ have a common intersection.*

▶ **Corollary 18.** *Let P be a set of $2n$ (uncolored) points in the plane, and let $\{(a_i, b_i), i = 1, \ldots, n\}$ be a max-sum matching of P. Then, there exists a point o of the plane such that for all $i \in \{1, \ldots, n\}$ we have:*

$$\|a_i - o\| + \|b_i - o\| \ \leq \ \sqrt{2} \cdot \|a_i - b_i\|.$$

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