Markov Renewal Theorem in the Series Scheme

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Abstract. The basic statements of the classical renewal theory are extended to the so-called Markov renewal equation. As a result of this extension the proof of the Markov renewal theorems for the scheme of series is given.

Key words: Markov renewal equation.

Introduction.

The classical renewal theory deals with the asymptotic properties of the solutions to the renewal equation

$$f(t) = g(t) + \int_0^t G(du) f(t-u),$$

where f is the function to be found, g is given function, and $G(du)$ is a given probability distribution. The classical renewal theorems describe the asymptotic properties of convolutions

$$f(t) = U * g(t) = \int_0^t U(du) g(t-u)\,, as\ t \to \infty,$$

where U is the potential of a homogeneous critical kernel $G(du)$, which is an ordinary probability distribution.

The basic statements of the classical renewal theory can be extended to the so-called Markov renewal equation

$$f(x,t) = g(x,t) + \int_E \int_0^t G(x, dy \times du)\, f(y, t-u), t \ge 0, x \in E.$$

where E is a given phase space, $G(x, dy \times du)$ is so-called semi-Markov kernel, $g(x,t)$ is a given function of $x \in E$, and $t \ge 0$, and $f(x,t)$ is the function to be found. Its solution is the convolution

$$f(x,t) = U * g(x,t) = \int_E \int_0^t U(x, dy \times du)\, g(y, t-u), t \ge 0, x \in E.$$

where $U(x, dy \times du)$ is the potential of the semi-homogeneous kernel $G(x, dy \times du)$.

Generally, the renewal theory has wide range of applications in mathematical practice. Markov renewal theorems are an analytical tool for studying the limiting behavior of Markov and related processes, including semi-Markov and regenerative processes.

Main results.

Let $(E, \mathfrak{B})$ be a measurable (phase) space with the countably generated σ-algebra $\mathfrak{B}$. We will assume, without loss of generality, that σ-algebra $\mathfrak{B}$ contains all one-point sets. Let us introduce a family of non-negative semi-homogeneous [3] kernels $G_\varepsilon(x, dy \times dt)$ which depend on a small parameter $\varepsilon > 0$.

Consider the Markov renewal equation

$$f_\varepsilon(x,t) = g_\varepsilon(x,t) + \int_E \int_0^t G_\varepsilon(x, dy \times du)\, f_\varepsilon(y, t-u), t \ge 0, x \in E, \qquad (1)$$

where $g_\varepsilon(x,t)$ is a given nonnegative $\mathfrak{B}\times\mathfrak{B}_+$-measurable function, $f_\varepsilon(x,t)$ is the function to be found, $\mathfrak{B}_+$ is the Borel σ-algebra on $R_+ = [0,\infty)$.

Next, we impose a number of restrictions. Let's assume that the kernels $G_\varepsilon(x,\{x\}\times dt)$ for all $x\in E$, converge to a probabilistic right-continuous function $F(x,dt)$ which measurably depends on all $x\in E$, in that sense

$$\left|\int_0^\infty G_\varepsilon(x,\{x\}\times dt)\varphi(t) - \int_0^\infty F(x,dt)\varphi(t)\right| \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{2}$$

for an arbitrary continuous bounded function $\varphi(t), t\geq 0$. It follows that for all $x\in E$

$$\lim_{\varepsilon\to 0} G_\varepsilon(x,\{x\}) = 1, \tag{3}$$

where $G_\varepsilon(x,\{x\})$ is the basis of the kernel $G_\varepsilon(x,\{x\}\times dt)$, that is $G_\varepsilon(x,\{x\}) = G_\varepsilon(x,\{x\}\times[0,\infty))$.

Denote the basis of the kernel $G_\varepsilon(x,dy\times dt)$ by $G_\varepsilon(x,dy)$ and let

$$\lim_{\varepsilon\to 0} G_\varepsilon(x,E\setminus\{x\}) = 0, \tag{4}$$

Suppose there exists a function $c(x)$ and a kernel $C(x,A)$ on $(E,\mathfrak{B})$ such that for all $x\in E$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{1-G_\varepsilon(x,\{x\})\} = c(x), \tag{5}$$

$$\sup_{A\in\mathfrak{B}}\left|\frac{1}{\varepsilon}G_\varepsilon(x,A) - C(x,A)\right| \underset{\varepsilon\to 0}{\longrightarrow} 0, x\notin A. \tag{6}$$

For convenience, we put $C(x,\{x\}) = 0, x\in E$. Note that based on (5) and (6) for all $x\in E$

$$|c(x)|<\infty, C(x,E)<\infty.$$

Let's demand that

$$\sup_{x\in E}|c(x)|<\infty, \sup_{x\in E} C(x,E)<\infty. \tag{7}$$

We will assume that

$$\sup_{\varepsilon>0}\int_T^\infty G_\varepsilon(x,E\times dt)t \underset{T\to\infty}{\longrightarrow} 0, x\in E. \tag{8}$$

From this, in particular, it follows that

$$\int_0^\infty F(x,dt)t<\infty, x\in E.$$

Denote by $m(x)=\int_0^\infty F(x,dt)t$ and finally assume

$$\inf_{x\in E} m(x) > 0. \tag{9}$$

W. Feller introduced the very important notion of direct Riemann integrability.

Namely, a family of functions $g_\varepsilon(x,t)$ on $E\times R_+$, that depend on a small parameter $\varepsilon>0$, is called directly Riemann-integrable if the series

$$\sum_{k=0}^\infty \sup_{k\leq t\leq k+1} g_\varepsilon(x,t) \tag{10}$$

$$\sup_{\varepsilon>0} \delta \sum_{k=0}^{\infty} \left\{ \sup_{k\delta \le t \le k\delta+\delta} g_\varepsilon(x,t) - \inf_{k\delta \le t \le k\delta+\delta} g_\varepsilon(x,t) \right\} \underset{\delta\to 0}{\longrightarrow} 0. \quad (11)$$

Under these conditions, the improper integral

$$\int_0^\infty g_\varepsilon(x,t)dt$$

is the limit of the integral sums constructed for a direct partition (hence the name) of the semi-axis $[0,\infty)$ uniformly on $\varepsilon > 0$ for all $x \in E$, that is

$$\sup_{\varepsilon>0} \left| \int_0^\infty g_\varepsilon(x,t)dt - \delta \sum_{k=0}^{\infty} g_\varepsilon(x,t_k) \right| \underset{\delta\to 0}{\longrightarrow} 0,$$

where $t_k \in [k\delta, k\delta+\delta]$, in contrast to the improper Riemann integral as limit of integrals over finite intervals.
That is why such a function $g_\varepsilon(x,t)$ is called directly Riemann-integrable.

Let the distribution function $F(x,dt)$be non-lattice for all $x \in E$ and there be a limit

$$\lim_{\varepsilon\to 0} \int_0^\infty g_\varepsilon(x,t)dt = \int_0^\infty g(x,t)dt = d(x), \ \ x \in E. \quad (12)$$

Each kernel $K(x,A)$ naturally generates a linear operator K that operates in Banach space $\boldsymbol{B}$ bounded $\mathfrak{B}$-measurable function f with a norm $\|f\| = \sup_{x\in E}|f(x)|$ by a formula

$$Kf(x) = \int_E K(x,dy)f(y).$$

Denote by M and D the operators corresponding to the kernels $m(x)$ and

$$D(x,A) = -c(x)I(x,A) + C(x,A).$$

Thus we have proved the following theorem.

Theorem. Let in conditions (2), (5), (6), (7), (8), (9),(10),(11),(12) for all $x \in E$ the probability distribution $F(x,t)$be non-lattice, then

$$\lim_{\substack{\varepsilon\to 0 \\ t\to\infty \\ \varepsilon t\to u}} f_\varepsilon(x,t) = e^{u\frac{D}{M}} M^{-1} d(x)$$

for all $x \in E$.

Conclusion.

The asymptotics of the solution of the Markov renewal equation when the basis $G_\varepsilon(x,dy) = G_\varepsilon\big(x, dy \times [0,\infty)\big)$ of the kernel $G_\varepsilon(x, dy \times dt)$close to the singular kernel $I(x,dy)$ on a given measurable phase space $(E,\mathfrak{B})$ was studied in [2].The main result of that study was formulated in the form of a theorem. At the same time, severe restrictions were imposed. Uniform convergence on $x \in E$ was required. In this paper, we prove a similar statement under weaker assumptions, namely, it is sufficient that the conditions of the theorem are satisfied for all $x \in E$. For this, a completely different idea of proof is used.

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