Hamiltonian Meander Paths and Cycles on Bichromatic Point Sets

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Abstract

We show that any set of n blue and n red points on a line admits a plane meander path, that is, a crossing-free spanning path that passes across the line on red and blue points in alternation. For meander cycles, we derive tight bounds on the minimum number of necessary crossings which depend on the coloring of the points. Finally, we provide some relations for the number of plane meander paths.

1 Introduction

Let S be a set of n red and n blue points on the x-axis. We call S a *bichromatic point set*. A *meander path* (*cycle*) P (C) is a spanning path (cycle) that visits the red and blue points of S in alternating order and the edges of P (C) also lie above and below the x-axis in alternation. We assume that every edge is a simple Jordan arc. If the edges of the path (cycle) do not cross each other, then we call it a plane meander path (cycle). See Figure 1 for examples. We label a bichromatic point set S by $p_1, \dots, p_{2n}$ from left to right. Unless stated otherwise, we direct a meander path to start at a blue point going upwards. We immediately observe the following.

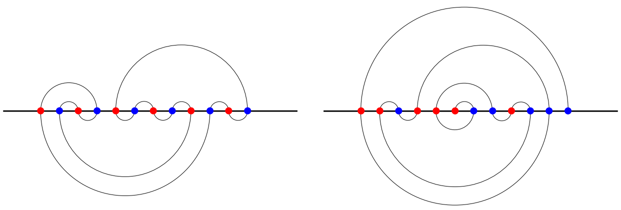

Figure 1: A plane meander cycle (left) and a plane meander path (right).

Proposition 1 *A plane meander cycle on a bichromatic point set exists if and only if the vertices are colored in alternating order along the x-axis.*

Proof. The sufficiency of the condition is evident for any n. To see that it is also necessary, assume w.l.o.g. that the leftmost point is red and orient the cycle so that it goes upwards at this point. As every cycle C alternates in both the color of the points and the location w.r.t. the x-axis, C goes upwards at every red point and downwards at every blue point. Moreover, C splits the plane into two connected regions, the interior region and the exterior region. Thus the parts of the x-axis which lie in the interior of C are bounded by a red point on the left and a blue point on the right. As the x-axis is split into alternating parts interior and exterior of C by the points, consecutive points on the x-axis cannot have the same color. □

Several questions arise from this first observation.

1. Which colorings yield plane meander paths?
2. Can we bound the number of necessary crossings of meandering cycles for a given coloring?
3. How many plane meander cycles / paths do exist?

For (1) we show the surprising result that any coloring admits a plane meander path. For (2) we present a tight lower bound for the number of crossings in any cycle in terms of how much the coloring differs from an alternating coloring. We conclude with a result on counting meander paths and some open problems.

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Related work. Meanders were already studied by Poincaré [21]. According to Lando and Zvonkin [17], the term "meander" was first suggested by Arnold [4], due to the analogy of the concept with a river starting from the north-west and meandering back and forth across an infinite horizontal road. Plane meander paths and cycles for uncolored point sets have been studied in various areas of mathematics and physics, including the map-folding problem [19], the stamp-folding problem [9, 15], polymer chains [10], and problems in differential geometry [21] or topology [4].

A challenging problem in the combinatorics of meanders is to estimate M_n, the number of plane meander cycles that go across a horizontal line $2n$ times. A lot of work has been done to enumerate plane meander cycles and paths [3, 5, 9, 11, 12, 14, 18, 22] and some of their variants [7, 13, 20]. It is conjectured that $M_n \sim Cr^n n^{-k}$, where C, r, and k are constants. The best current bounds for r are $11.380 \leq r \leq 12.901$, due to Albert and Paterson [3], and it is believed that $k = \frac{29+\sqrt{145}}{12} \approx 3.42$ [11]. See also sequence A005315 in the on-line encyclopedia of integer sequences (`https://oeis.org/`) for further references.

For bichromatic point sets, the study of the existence of plane meander cycles and paths is related to the following classical problem in computational geometry (cf. [2] and see also [16]): What is the largest number $s(n)$ such that, for every set of n red and n blue points on a circle, there exists a noncrossing straight-line alternating path consisting of $s(n)$ points? It has been shown that there exist configurations of red and blue points on the circle such that $s(n) \leq \frac{4}{3}n + o(n)$ [1, 16], and it is conjectured that $|s(n) - \frac{4}{3}n| = o(n)$ for any bichromatic point set.

If straight-line segments are replaced by simple Jordan arcs, and the arcs alternate inside and outside the circle, then the previous problem transforms into determining the largest number $s'(n)$ such that every set of n red and n blue points on a line (one can think of a line as a circle of infinite radius) admits a plane meander path on $s'(n)$ of the points. In contrast to the result for noncrossing straight-line alternating paths, we show the surprising result that $s'(n) = 2n$, that is, any valid coloring admits a plane meander path.

When crossings are allowed, we present a tight lower bound for the number of crossings in any meander cycle in terms of of how much the coloring differs from the alternating coloring on the line. A similar result was obtained in [6], where a tight lower bound was given for the number of crossings in any straight-line Hamiltonian alternating cycle for bichromatic point sets on the circle.

2 Plane paths and crossing-minimal cycles

To show that for any valid coloring of S there exists a plane meander path we developed an incremental approach (see the full version). An alternative version is used in this section, where we construct plane meander paths based on a tree structure determined by the coloring of the point set. It turns out that the depth of such a tree gives a tight lower bound on the number of crossings in any meander cycle on the set. We denote the set of consecutive points on the x-axis from p_i to p_j by $[i, j]$. Let $r_S(i, j)$ and $b_S(i, j)$ be the number of red points and the number of blue points of S in the interval $[i, j]$, respectively. Let $d_S(i, j) = r_S(i, j) - b_S(i, j)$, and let $\Delta(S)$ be the maximum of the absolute values of these differences over all intervals $[i, j]$, that is, $\Delta(S) = \max_{[i,j]} |d_S(i, j)|$. The next observation allows us to cyclically shift the point configuration.

Observation 2 *For any bichomratic point set S and any plane meander path P (cycle C) on S, P (C) is also a plane meander path (cycle) of any point set S' that is obtained by successively moving the first i points of S after the last point of S.*

Theorem 3 *On every bichromatic point set, there exists a plane meander path and a meander cycle with $\Delta(S) - 1$ crossings.*

Proof. By Observation 2, we can assume that $d_S(1, j) = \Delta(S)$ for some $1 \leq j \leq n$ and hence $r(1, j) > b(1, j)$ (see Figure 2a). We assign a left bracket to every red point and a right bracket to every blue point, as shown in Figure 2b. The number of left brackets in an interval $[1, i]$ is always at least the number of right brackets. (Otherwise, if this number is less than the number of right brackets in $[1, i]$, then $\Delta(S) < r_S(i + 1, j) - b_S(i + 1, j)$ when $i < j$ and $\Delta(S) < b_S(j + 1, i) - r_S(j + 1, i)$ when $i > j$, a contradiction).

We now connect the brackets with arcs. If a right bracket directly follows a left bracket, we connect then with an arc drawn above the x-axis. We then remove these two brackets and repeat the process of connecting two consecutive brackets (the first to the left and the second to the right) until all brackets have been connected with arcs (see Figure 2c).

Observe that this set of arcs form a tree. Each arc can be seen as a vertex of the tree and two vertices are connected if their corresponding arcs see each other. We also add an extra vertex on the top part as the root of the tree (see Figure 2d). Also observe that the depth of the tree is precisely $\Delta(S)$.

We say that all vertices (arcs) that are at distance k from the root form the k-th level. Next, we connect all arcs with the same level k to form a plane path P_k. To that end, assume that the arcs $a_1, a_2, \ldots, a_j$ of level k are ordered from left to right. For $1 \leq i \leq j - 1$, we connect the second endpoint of arc a_i to the first endpoint of arc a_{i+1} using an arc below the x-axis

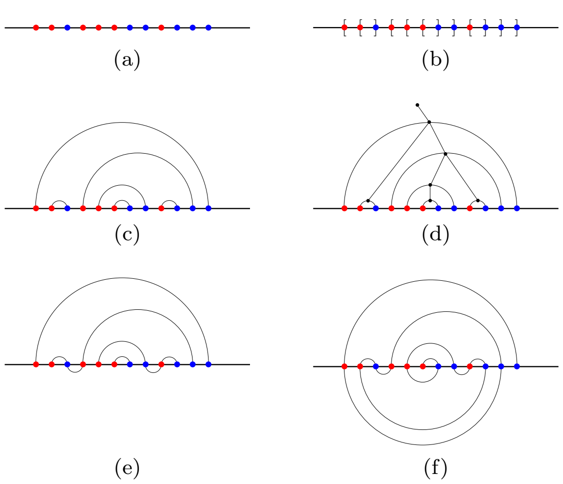

Figure 2: Building a noncrossing meander path.

(see Figure 2e). Repeating this process for all levels, we obtain $\Delta(S)$ paths whose union is crossing-free and covers all points in S. (The paths may also be completed to $\Delta(S)$ disjoint plane meander cycles.)

From this set of paths, we obtain a plane meander path on S as follows. Note that, from left to right, the first endpoints of these paths are all red and the last endpoints are all blue. Thus, these paths can be connected in a spiraling way by $\Delta(S) - 1$ edges (between the red/blue endpoint of each path P_k and the blue/red endpoint of the next path P_{k+1}) to a plane meander path P on S (see Figure 2f). Finally, observe that by connecting the two endpoints of P, we obtain a meander cycle with exactly $\Delta(S) - 1$ crossings. $\square$

In the above proof we actually have two options for the order in which we connect the paths of the different levels, yielding two different meander paths or (crossing) meander cycles. We note that the construction can be done in linear time, by first finding the interval giving $\Delta(S)$ (as described in [8]), and then building the tree using a stack.

It turns out that a meander cycle constructed in the described way is actually crossing-minimal, i.e., the number of crossings of a meander cycle C on S is at least $\Delta(S) - 1$. To prove the bound, we use the following trivial observation.

Observation 4 *Let C_1 and C_2 be two meander cycles on two bicolored point sets S_1 and S_2, with $S_1 \cap S_2 = \emptyset$. Suppose that edge $uv \in C_1$ crosses edge $u'v' \in C_2$, with u and u' being red and v and v' being blue. Then, replacing edges uv and $u'v'$ by edges uv' and $u'v$ as shown in Figure 3, we obtain a meander cycle on $S_1 \cup S_2$.*

Theorem 5 *Given a bicolored point set S and a meander cycle C on S, the number of crossings of C is at least $\Delta(S) - 1$.*

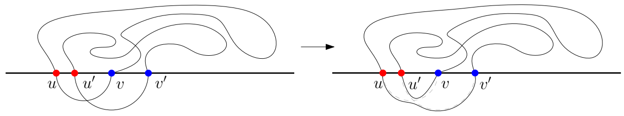

Figure 3: Obtaining a meander cycle on $S_1 \cup S_2$ by combining two meander cycles on S_1 and S_2.

Proof. The proof is by induction on the number of points. The base of the induction is the bicolored point set S consisting of one red point and one blue point, and the cycle consisting of the duplicated edge connecting these two points, one of the edges drawn above the x-axis and the other below the x-axis.

Let C be a meander cycle on a bicolored point set S with $2n$ points. If C does not have any crossing, then the points of S alternate along the x-axis. Hence, $\Delta(S)$ is trivially 1 and the theorem follows.

Assume then that C has crossings and take one of them, the crossing defined by edges uv and $u'v'$, being u and u' red and v and v' being blue. We replace these two edges by edges uv' and $u'v$ as shown in Figure 4 (note that there are two ways of drawing uv' and $u'v$, depending on the relative positions of u, u', v and v'). In this way, we obtain either a new meander cycle C' on S (with one fewer crossings) or two meander cycles C_1 and C_2 on two disjoint bicolored points sets S_1 and S_2. By Observation 4, if these two cycles cross, then we can obtain a new meander cycle on S (with two fewer crossings). By iterating this process on the crossings of C, we obtain either a noncrossing meander cycle on S, or two meander cycles C_1 and C_2 on two disjoint bicolored points sets S_1 and S_2 such that C_1 and C_2 do not cross.

In the first case, the points of S must alternate along the x-axis, so $\Delta(S) = 1$ and the theorem obviously holds. In the second case, we apply induction on C_1 and C_2. As C_1 and C_2 do not cross, the number of crossings of C is at least $\Delta(S_1) - 1 + \Delta(S_2) - 1 + t$, where t is the number of crossings of C removed to obtain C_1 and C_2. Observe now that, if $[i, j]$ is the interval such that $d_S(i, j) = \Delta(S)$, then $d_S(i, j) = r_S(i, j) - b_S(i, j) = (r_S(i, j) \cap S_1) + (r_S(i, j) \cap S_2) - (b_S(i, j) \cap S_1) - (b_S(i, j) \cap S_2) \leq \Delta(S_1) + \Delta(S_2)$. As $t \geq 1$, the number of crossings of C is at least $\Delta(S_1) - 1 + \Delta(S_2) - 1 + t \geq \Delta(S) - 1$, as required. $\square$

3 Final remarks

Our construction of plane meander paths shows that every bichromatic point set admits at least two meander paths. In our attempt of bounding the number of paths, we showed that for some point configurations the number of paths is determined by the number of the number of *blocks*, i.e., maximal consecutive monochromatic subsets.

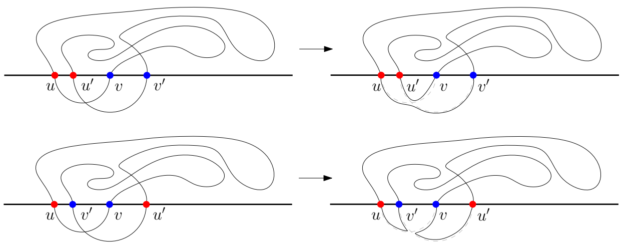

Figure 4: Replacing edges uv and $u'v'$ by edges uv' and $u'v$.

Theorem 6 *Let S and S' be two bichromatic point sets such that all the blocks have size $s \geq 2$ and $s' \geq 2$, respectively, and both sets have the same number k of blocks (i.e., $|S| = k \cdot s$ and $|S'| = k \cdot s'$). Then S and S' have the same number of meander paths.*

To prove this theorem, we show several structural properties for meander paths on S: The vertices of a block occur along the path from left to right or from right to left; in the *top matching* induced by the path, every block is completely matched to some other block; and in the *bottom matching*, every block shares at least $s-1$ edges with some other block. The proof is deferred to the full version of this work.

We conclude with some open problems.

- The lower bound on the minimal number of meander paths in terms of the number of vertices is constant, due to the example with two blocks. Can we get a lower bound in terms of the number of blocks?
- What is the number of meander paths for bichromatic point sets with uniform block size s and k blocks? For $s \geq 2$, it only depends on k.
- What about three colors? It is not always possible to have a meander path (take three different blocks of the same size). What would be the length of a maximal, not necessarily spanning, properly colored path whose edges are alternatingly above and below the x-axis? What is the complexity of deciding whether there is a spanning path, or a path of length k?
- Can we find a reasonable flip operation between meander paths of a given bichromatic point set? What would be the properties of the resulting flip graph?
- How fast can we count the number of meander paths for a given bichromatic point set?

References

[1] M. Abellanas, A. García, F. Hurtado, and J. Tejel. Caminos alternantes. In *Actas X Encuentros de Geometría Computacional*, pages 7–12. In Spanish.

[2] J. Akiyama and J. Urrutia. Simple alternating path problem. *Discrete Math.*, 84(1):101–103, 1990.

[3] M. H. Albert and M. S. Paterson. Bounds for the growth rate of meander numbers. *J. Comb. Theory, Ser. A*, 112(2):250–262, 2005.

[4] V. Arnold. The branched covering of $cp^2 \to s^4$, hyperbolicity and projective topology. *Sib. Math. J.*, 29:717–726, 1988.

[5] B. A. Bobier and J. Sawada. A fast algorithm to generate open meandric systems and meanders. *ACM Trans. Algorithms*, 6(2):42:1–42:12, 2010.

[6] M. Claverol, A. G. Olaverri, D. Garijo, C. Seara, and J. Tejel. On Hamiltonian alternating cycles and paths. *Comput. Geom.*, 68:146–166, 2018.

[7] V. Delecroix, E. Goujard, P. Zograf, and A. Zorich. Enumeration of meanders and Masur-Veech volumes. arXiv e-print arXiv:1705.05190v2, 2019.

[8] D. P. Dobkin, D. Gunopulos, and W. Maass. Computing the maximum bichromatic discrepancy with applications to computer graphics and machine learning. *J. Comput. Syst. Sci.*, 52(3):453–470, 1996.

[9] P. Francesco, O. Golinelli, and E. Guitter. Meanders: A direct enumeration approach. *Nuc. Phys. B*, 482:497–535, 1996.

[10] P. Francesco, O. Golinelli, and E. Guitter. Meander, folding, and arch statistics. *Mathl. Comput. Model.*, 26:97–147, 1997.

[11] P. Francesco, O. Golinelli, and E. Guitter. Meanders: exact asymptotics. *Nucl. Phys.*, 570:699–712, 2000.

[12] R. Franz and B. Earnshaw. A constructive enumeration of meanders. *Ann. Combinatorics*, 6:7–17, 2002.

[13] M. Fukuda and I. Nechita. Enumerating meandric systems with large number of loops. *Ann. Henry Poincaré.* To appear. Preliminary version: arXiv:1609:02756v2, 2016.

[14] I. Jensen. A transfer matrix approach to the enumeration of plane meanders. *J. Phys. A*, 33:5953–5963, 2000.

[15] J. Koehler. Folding a strip of stamps. *J. Combinat. Theory*, 5:135–152, 1968.

[16] J. Kynčl, J. Pach, and G. Tóth. Long alternating paths in bicolored point sets. *Discrete Math.*, 308(19):4315–4321, 2008.

[17] S. K. Lando and A. K. Zvonkin. Plane and projective meanders. *Theor. Comput. Sci.*, 117:227–241, 1993.

[18] S. Legendre. Foldings and meanders. *Australas. J. Comb.*, 58:275–291, 2014.

[19] W. Lunnon. A map folding problem. *Math. Computat.*, 22:193–199, 1968.

[20] A. Panayotopoulos and P. M. Vlamos. Partitioning the meandering curves. *Math. in Comp. Sci.*, 9(3):355–364, 2015.

[21] E. Poincaré. Sur un théorème de géométrie. *Rend. Circ. Mat. Palermo*, 33, 1912.

[22] J. Sawada and R. Li. Stamp foldings, semi-meanders, and open meanders: Fast generation algorithms. *Electr. J. Comb.*, 19(2):P43, 2012.