An analytical solution to the problem of bending-torsional vibrations of a thin-walled rod

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Abstract. The article provides a brief overview of the works devoted to the stability and bending-torsional vibrations of thin-walled rods. The problem is formulated and an analytical solution to the problem of bending-torsional vibrations of a thin-walled rod is given. The analysis of the obtained solutions is given.

1. Introduction

Common mathematical interpretation makes related problems of free oscillations and elastic stability of rod systems. These two problems developed independently for some time, until the similarity of eigenvalue problems in stability theory and oscillation theory attracted the attention of a number of researchers. In 1937, the work of P. F. Papkovich appeared [1], which for the first time indicated the commonality of these two problems. Then there are works such as [2,3,4,5,6,7,8]. By that time, in the theory of stability in the stage of development and improvement were such general methods as the method of forces, the method of displacement, the mixed method. The method of initial parameters and some others, allowing to calculate both individual rods and rod systems. One of the most convenient was the method of displacement, the canonical form of which is given in [6]. The problem was reduced to a homogeneous system of homogeneous transcendental equations with a large number of unknown parameters, which caused difficulties of a quantitative nature and contributed to the emergence of works in which various methods of approximate calculation [9,10,11,12] for stability were carried out, and such methods as the Rayleigh method, the Ritz method, the Bubnov-Galerkin method for calculating. In all works, the loss of stability or free oscillations occur in a bending form, since the rod is described by the Bernoulli-Euler equation. However, even a centrally compressed rod under asymmetric boundary conditions can lose stability or oscillate in a torsional-Flexural form. In this case, there is a need for joint consideration of homogeneous equilibrium equations or free oscillations of the rod in space. In stability, this problem was first solved by L. Prandtl [13]; a new section appeared-the stability of the plane form of deformation. A generalization of the Euler problem and the Prandtl problem was found in S. P. Timoshenko [14] on the stability of an off-center compressed band. The first problem of stability of a centrally compressed strip with continuous reinforcement along a line parallel to the Central axis is given in V. Z. Vlasov [15]. In the early 80s of the XX century, a number of works [16,17,18] related to Flexural-torsional forms of loss of stability of centrally compressed rods were published, in which the method of displacement was applied. Increasing the cross-sectional height of flat bar structures is a natural way to increase the load-bearing capacity and reduce material consumption. This brings nature closer to the Prandtl model, and the question arises of ensuring the stability of such structures in the plane of least stiffness. It is known that stability is provided by the statement of connections. These connections, as a rule, due to design solutions are placed not on the axis of the element, which stimulates the bending-torsional form of loss of stability. Until the 80s of the XX century there was no analytical method for calculating such structures for stability from the plane, and the stability assessment was made approximately. In 1982-1983 laboratory of stability of TSNIISK. Kucherenko in accordance with the theme of the target program of Gosstroy of the USSR developed recommendations for improving the methods of calculation of wooden structures (cipher OTS.031.055.I6C), in which a special form of the displacement method was proposed and developed for the problems of spatial stability of flat rod structures with non-axial point reinforcements from the plane. The capabilities of the device suggest its use for the calculation of structures made of any material, and the table of expressions of reactive forces of the elements of the main system makes it simple for engineering treatment. As you know, a kind of "passport" design in static calculation is the critical force, dynamic-the spectrum of free oscillations. And if the problem of finding eigenvalues in statics is solved, the solution of the problem in dynamics requires the development of an effective calculation method for finding eigenfrequencies and waveforms, the study of the influence of various parameters of the system on the fundamental frequencies.

2. Methods

In solving the dynamic problem, we will proceed from the D'Alembert principle, considering the dynamics problem formally, as static with the addition of inertia forces to the elastic forces. Given the relationship of dynamic problems and stability problems, we write the differential equations of spatial bending-torsional oscillations of a single rod by adding inertial terms to the stability equations. These equations are derived in [15] and have the form:

$$\begin{cases} EI_y\,\xi^{IV} + P\xi^{II} - m\omega^2\xi = 0; \\ (GJd - Pr^2)\,\theta^{II} + mr^2\omega^2\;\theta = 0, \end{cases} \tag{1}$$

where EI_y, GJ_d – the cruelty of the cross section, respectively, for bending and torsion;

P- longitudinal force;

ξ- the deflection amplitude of the rod axis in the x-axis direction;

θ- the amplitude of the twist angle of the cross section relative to the z axis;

$$r = \sqrt{r_x^2 + r_y^2}\;;$$

r_x^2, r_y^2 – radii of inertia of cross section;

Linear mass of the rod:

$$m = \frac{\gamma A}{g}$$

γ- density of rod material;

A- cross-sectional area of the rod;

g – acceleration of gravity;

ω – frequency of free oscillations.

The coordinate axes are shown in Fig.1.

Equations (1) are valid under the following assumptions: - the rod material is perfectly elastic; - when the axis of the rod is curved, the sections remain flat; - during torsion, the cross section is not distorted; - the cross section of the rod is constant in length; - cross load in the ZOY plane is applied at the axis level; - the cross section of the rod is narrow:

$$EI_y \ll EI_x\;;\; GJd \ll EI_x$$

Equations (1) are a special case of equations of spatial bending-torsional vibrations of a thin-walled rod loaded with a longitudinal force.

If in equations (1) the longitudinal force is considered as a given function of time t: P = P (t), then these equations will relate to the general theory of spatial dynamic stability of prismatic rods.

In the second equation of system (1), in contrast to the usual "beam" theory, there is an additional term $Pr^2\theta^{II}$, its occurrence is associated with a fundamentally different idea of the nature of the application of the force P to the end section of the rod in the theory of V.Z. Vlasov [1,2,3]. According to the "beam" theory, the force P is applied to the axis of the rod (Fig. 2), and if purely torsional vibrations are disturbed, they will not work, since the length of the axis of the rod remains constant. In fact, the force P is the resultant diagram of the pressure applied to the end face of the rod (Fig. 3) and is an abstract concept. For the case in Fig. 3, all the fibers of the rod, when purely torsional vibrations are perturbed, turn into helical lines whose projections on the central axis are smaller than their initial length, therefore, the stresses applied to the end of the rod do elementary work, the integral of which on the area of the end gives the work of the external force P. The solution of the second equation of the system will be written:

$$\theta(z) = c_1 \sin nz + c_2 \cos nz \qquad (2)$$

Where:

$$n^2 = \frac{mr^2\omega^2}{GJd - Pr^2} \qquad (3)$$

Let one end of the rod (*z =0*) be rigidly fixed and the other (*z = l*) be free, then the boundary conditions are expressed by equalities:

$$\theta(0) = 0;\ \theta'_{(0)} = 0. \qquad (4)$$

From expression (4) we find arbitrary constants c_1 and c_2 of the general solution (2), we have:

$$c_2 = 0;\ c_1 \cos nl = 0.$$

The constant c_1 cannot be equal to zero, because otherwise $\theta(z) \equiv 0$. A non-trivial solution is obtained if:

$$n_K = (2K - 1)\pi/2l. \qquad (5)$$

Substituting (3) in (5), we have:

$$\frac{mr^2\omega^2}{GJd - Pr^2} = \frac{(2K-1)^2\pi^2}{4l^2} \qquad (6)$$

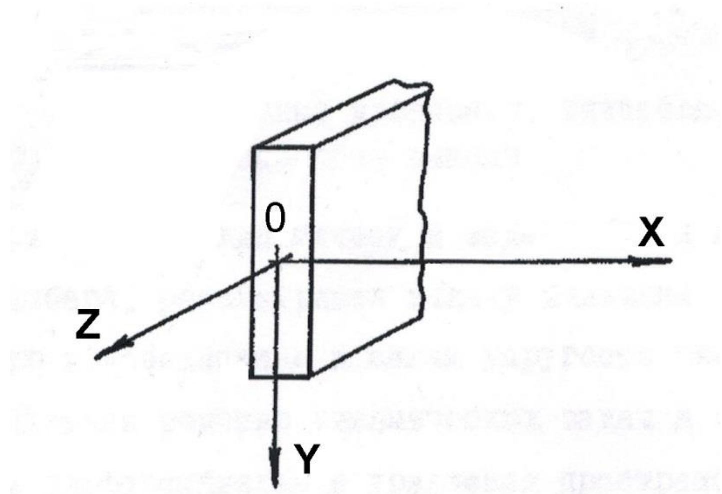

Fig.1. Coordinate axes

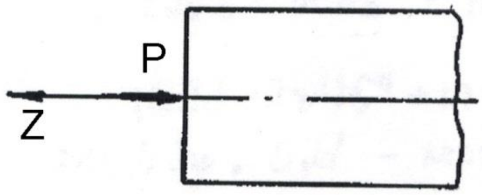

Fig.2 Compressive force P

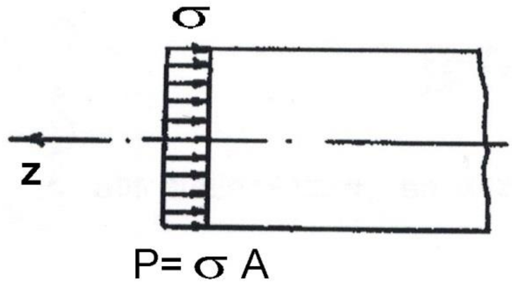

Fig.3. Pressure plot

3. Results

From expression (6) we obtain the value of the critical force:

$$P = \frac{GJd}{r^2} - \frac{4m\omega^2 l^2}{(2K-1)^2\pi^2}, \tag{7}$$

where K - the frequency number.

$$\omega^2 = \frac{(P_{\text{кр}} - P)(2K-1)^2\pi^2}{4ml^2}; \quad P_{\text{кр}} = \frac{GJd}{r^2} \tag{8}$$

4. Conclusions

The analysis of equations (7) and (8) allows to obtain a qualitative result: with the growth of the longitudinal force, the resistance of the rod to twisting and the frequency of free torsional oscillations decreases.

The solution of equation (1) can be represented as:

$$\begin{cases} \xi = c_1 \sin\lambda_1\eta + c_1 \cos\lambda_1\eta + c_3 \sinh\lambda_2\eta + c_2 \cosh\lambda_2\eta; \\ \theta = c_5 \sin\lambda_3\eta + c_6 \cos\lambda_3\eta. \end{cases} \tag{9}$$

In (9), the notation:

$$\lambda_1^2 = \frac{v^2}{2} + \sqrt{\frac{v^4}{4} + u^4};$$

$$\lambda_2^2 = -\frac{\nu^2}{2} + \sqrt{\frac{\nu^4}{4} + u^4};$$

$$\lambda_3^2 = \frac{mr^2\omega^2 l^2}{(GId - Pr^2)} = \frac{u^4}{(12\frac{n^2}{S} - \nu^2)};$$

$$\nu^2 = \frac{Pl^2}{EI_y};\ u^4 = \frac{m\omega^2 l^4}{EI_y};\ S = \frac{EI_y}{GId};\ n = \frac{l}{h};$$

$$\eta = \frac{z}{l}$$

η - relative coordinate,
l - the length of the rod,
h - the height of the rod,
E - the elastic modulus of the rod material,
G - shear modulus,
I_y - moment of inertia of the section relative to the y axis,
I_d - sectorial moment of inertia,
z - the current coordinate.

Six arbitrary constants $c_1 ... c_6$ solutions (9) correspond to six (three at each end of the rod) boundary conditions. Solutions (9) are suitable for solving any boundary value problems, since all static and kinematic variables can be expressed in terms of ξ and θ.

At first glance, the system (9) decays into two independent equations, since each contains only one variable, i.e. bending and torsion can be considered separately. In fact, under complex boundary conditions (there may be 12 combinations of boundary conditions at the two ends of the rod), arbitrary constant solutions (9) are interdependent, i.e., both bending and torsional vibrations are possible under central compression.

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