Processes for a Colony Solving the Best-of-N Problem using a Bipartite Graph Representation

Puneet Jain and Michael Goodrich

May 13, 2021

Processes for a Colony Solving the Best-of-N Problem using a Bipartite Graph Representation

Puneet Jain[0000−0002−6420−4999] and Michael A. Goodrich[0000−0002−2489−5705]

Department of Computer Science, Brigham Young University, Provo, UT, USA
puneetj@byu.edu, mike@cs.byu.edu

Abstract. Agent-based simulations and differential equation models have been used to analyze distributed solutions to the best-of-N problem. This paper shows that the best-of-N problem can be also solved using a graph-based formalism that abstractly represents (a) agents and solutions as vertices, (b) individual agent states as graph edges, and (c) agent state dynamics as edge creation (attachment) or deletion (detachment) between agent and solution. The paper identifies multiple candidate attachment and detachment processes from the literature, and then presents a comparative study of how well various processes perform on the best-of-N problem. Results not only identify promising attachment and detachment processes but also identify model parameters that provide probable convergence to the best solution. Finally, processes are identified that may be suitable for the best-M-of-N problem.

1 Introduction

Swarms and *colonies* are important ways to organize bio-inspired agents [3, 38, 18, 44, 41, 20]. Varying Brambilla et al.'s swarm taxonomy [4], spatial swarms are characterized by persistently colocated agents. Hub-based colonies, by contrast, include agents colocated at a hub and spatially distributed agents with few inter-agent interactions. Hub-based colonies are important because they (a) can potentially include more agents than swarms [32, ch. 1] and (b) provide distributed solutions to the best-of-N and best-M-of-N problems [46, 41, 45, 47, 24].

Two approaches are often used to design and analyze hub-based colonies: *agent-based* (AB) models and *differential equation* (DE) models. AB models typically use state machines to generate agent behavior, which are often used in empirically oriented experiments to explore how various settings affect colony behavior [11, 46]. DE models include mean-field models [6, 7], evolutionary models [44, ch. 4], and certain kinds of probabilistic models [44, ch. 3]. DE models are often used to find theoretical properties such as stable attractors, bifurcations, and steady state distributions [34, 28, 26]. Combining AB and DE models can link micro-level agent behaviors to macro-level swarm phenomena [10].

For spatial swarms, *graph-based models* have proven useful for evaluating how group size, communication networks, and misinformation can affect the swarm. Both theoretical and empirical results can be derived for graph-based models, complementing the results provided by differential equation models [40, 31]. In contrast, graph-based approaches are rarely applied to hub-based colonies.

This paper details a dynamic bipartite graph-based model for a best-of-N problem. A mathematical formalism is outlined and empirical demonstrations are provided based on two fundamental processes: the probability that an agent will “attach” to a site (e.g., assess it, recruit to it) and “detach” from a site (e.g., return to the hub to rest, be recruited to assess another site). Different attachment and detachment processes are compared, including both homogeneous and heterogeneous behaviors. The merits and drawbacks of various processes are discussed and applied to the best-M-of-N problem.

2 Related Literature

A graph-based model of a hub-based colony was proposed in [10]. Unlike [10], which represented agent states as graph *vertices*, this paper represents agent states as graph *edges* between an agent and various points of interest in the world. Preliminary work in [19] describes a bipartite graph formalism with nominal “attachment” and “detachment” processes but fails to discuss why the attachment/detachment processes work or how they compare to other processes.

Preferential attachment [2, 36, 35] motivates the degree-based attachment process proposed below. Bipartite graph models, similar to the one used in this paper, have been used with preferential attachment to model virus spread [37]. Ant colony optimization with bipartite graphs has also been used for assigning cells to switches and for reconstructing newspaper articles [42, 16]. Preferential attachment has been used in bio-inspired applications to model animal decisions to join or leave a group [44, ch. 2], similar to the processes in the present paper in which agents cluster in a group around important sites.

A survey of best-of-N problems is presented in [46], and examples of the problem in dynamic environments are in [33, 39]. Best-of-N problems include selecting the best candidate for a job [8], selecting the best nest [41], finding the best foraging site [20], and finding the top nodes in social networks [48, 25].

3 Bipartite Graph Formalism

Best-of-N problems are usually characterized by two entities: *agents* and *solutions*. *Agents* actively explore the world, communicate with other agents, and participate in distributed decision making. Since many best-of-N solutions correspond to decisions about physical locations, we refer to solutions as *sites*.

Agents in both agent-based and differential equation models typically adopt one of several possible states. For example, honeybees finding a new nest can be in states that *explore* the world, *assess* a possible nest site, *dance* to advertise a possible nest site, *observe* other honeybees in the nest, *rest*, or *commit* to a nest site [41, 34, 9]. Similarly, consensus decision in various ant species include scout, forager, recruiter, and passive states [14, 20, 27]. Transitions between agent states depend on what other agents do as well as what is observed in the environment [14, 20, 41]. Agents, sites, and state transitions can be modeled as a bipartite graph.

Bipartite Graph Formalism. Let $G = (V, E)$ be the bipartite graph constructed by partitioning a set of vertices, V, into *agent* vertices and *site* vertices; $V = V_{\text{agent}} \cup V_{\text{site}}$ and $V_{\text{agent}} \cap V_{\text{site}} = \emptyset$. Since G is bipartite, the edge set consists only of edges connecting an agent vertex to a site vertex, $E = \{(a, s) | a \in V_{\text{agent}} \text{ and } s \in V_{\text{site}}\}$. Without loss of generality, each site is assigned a quality in the range $0 \leq \text{qual}(s) \leq 1$.

An edge between agent a and site s represents a subset of possible *site-oriented* states. For honeybee and ant colonies, site-oriented states include assess, dance, recruit, pipe, or commit states. An agent without an edge indicates that the agent is in a *site-agnostic* state; for honeybee and ant colonies, site-agnostic states include rest, observe, or explore states. Thus, the presence or absence of edges between an agent and a site are abstract representations of possible agent states. Agents in hub-based colonies typically assess, promote, or take action for only one site at a time, whence $\forall a \in V_{\text{agents}}, \deg(a) \leq 1$.

Graph Dynamics. There are multiple ways the $|V_{\text{agents}}|$ agents can be connected to the $|V_{\text{sites}}|$ sites. A specific graph is called a *configuration* and is denoted by x, and G_t denotes a time-indexed random variable of possible configurations. Graph dynamics model how agents transition between site-oriented and site-agnostic states. A transition from a site-agnostic state to a site-oriented state is represented by adding an edge, and vice versa by deleting an edge.

Adding (removing) a graph edge between an agent and a site is called *attachment* (*detachment*). The probabilities of attachment or detachment abstractly encode the nondeterminism in transitions between site-oriented and site-agnostic states. These probabilities induce a random process that maps one graph to another, $G_t \rightarrow G_{t+1}$. G_t and G_{t+1} can differ in at most two edges because attachment and detachment are independent and only affect one edge at a time. Time steps abstractly represent individual agent transitions between a site-oriented and a site-agnostic state rather than real-time estimates of colony behavior.

4 Attachment and Detachment Probabilities

This section uses patterns from biology to motivate attachment and detachment processes. Let A and R denote Attachment and Detachment random variables, respectively[1].

4.1 Attachment Probability

First, an agent a is *selected* with uniform probability $\frac{1}{|V_{\text{agents}}|}$. If $\deg(a) > 0$, the agent is already attached to a site, so no edge is added. Second, an edge (a, s) is potentially *added* using an attachment process below.

[1] R denotes edge removal/detachment so that D can be used to denote degree.

Motivation for Attachment Processes. Biological models of the best-of-N problem suggest that agents are more likely to connect to popular sites.

- The probability that an observing honeybee will be recruited to a site grows with the number of honeybees dancing for the site [41]
- The amount of attraction pheromone deposited on a trail grows with the number of ants following the trail [12]
- "[T]he number of nestmates [an ant] encounter[s] ... [acts] as a stimulus to switch ... to recruitment by carrying ... nestmates to a new nest" [14]
- Encountering more returning foragers stimulate more ants to forage [20].

Agents can vary in their ideal group size, which means that popularity-based aggregation can be modulated by *individual* preferences.

- The probability of a *Holocnemus pluchei* spider staying or leaving a shared web depends on how large and well fed the spider is [23].
- The group sizes of large mammalian herbivores might be affected by individual preferences [13, 17].
- Group formation models assume that individuals have sociality thresholds [5].
- Localized interactions can cause fish to differ in whether they join a shoal [21].
- Some ants prefer risky exploration to joining a majority [22].

Baseline Attachment. Given agent a, a site s is chosen with uniform probability $\frac{1}{|V_{\text{sites}}|}$ and an edge (a, s) is formed. This baseline provides insight into clustering driven solely by the detachment process.

Nominal Attachment. Popularity-based networks are well-modeled by *preferential attachment* [2][44, ch. 2]. Thus, nominal attachment probabilistically favors sites with higher degree.

Select Degree. There can be several different sites with the same degree. Let $\text{Deg}(x_t) = \{d : \exists s \in V_{\text{sites}} \text{ for which } \deg(s) = d\}$, be the set of unique degrees in configuration x_t. The monotonic function $f(d, G_t) = d + 1/|\text{Deg}(x_t)|$ encodes two factors that govern the probability that a degree is chosen: (1) higher degree sites should induce more agents to attach, and (2) sites with zero degrees should have a non-zero probability of attachment so that new sites can be "discovered". The probability of selecting a degree is obtained by sampling from the set $\deg(G)$ obtained by normalizing $f(\cdot)$, yielding $P_{D|G_t}(d|x_t) = f(d)/\big[\sum_{d' \in \text{Deg}(x_t)} f(d')\big]$, where D denotes the degree random variable

Select Site. Given the degree, a site is chosen with uniform probability from sites with that degree, $P_{S|D}(s|d, x_t) = 1/|\{s' : \deg(s') = d\}|$. The probability of sampling site s given the graph is derived using the chain rule and marginalizing,

$$P_{S|G_t}(s|x_t) = \sum_d P_{S|D,G_t}(s|d, x_t) P_{D|G_t}(d|x_t) = \frac{P_{D|G_t}(\deg(s)|x_t)}{|\{s' : \deg(s') = \deg(s)\}|}. \quad (1)$$

Add Edge. Let E denote the "add edge" random variable. No edge is added if the randomly selected agent is already attached. An edge is added to an unattached agent using the product of Eq. (1) and the uniform probability of selecting the agent, yielding $P_{E|G_t}\left[(a, s)|x_t\right] = \frac{P_{D|G_t}(\deg(s)|x_t)}{|V_{\text{agents}}| \cdot |\{s' : \deg(s') = \deg(s)\}|}$.

Saturated Degree Attachment. A variation of preferential attachment attaches agents to sites with probability (a) linearly proportional to the site's degree up to a specified saturation degree, d^{sat}, and (b) zero if $\deg(s) > d^{\text{sat}}$. This saturated attachment process uses the *Select Site* and *Add Edge* probabilities from the nominal attachment process, but uses the *Select Degree* probability,

$$P_{D|G_t}(d|x_t) = \begin{cases} \frac{f(d)}{\sum_{d' \in \text{Deg}(x_t), d' \leq d^{sat}} f(d')}, & d \leq d^{\text{sat}} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Preferred Degree Attachment. Variations of preferential attachment also allow individual agents to prefer different site degrees. Let d_i^* denote the preferred degree for agent a_i, and let $\Delta_{ij} = |d_i^* - \deg(s_j)|$ denote the difference between agent a_i's preferred degree and the degree of site s_j. When agents are allowed a degree preference, all agents may prefer the same degree (*homogeneous*) in which case $\forall i \; d_i^* = d^*$, or agents may prefer different degrees (*heterogeneous*). Section 5 specifies parameters for homogeneous and heterogeneous degree preferences.

Select Site. The monotonically decreasing function $f(\Delta_{ij}) = 1/(1 + \Delta_{ij}^2)$ favors sites with degree close to the preferred degree. Normalizing yields

$$P_{S|G_t}(s_j|x_t) = \frac{f(\Delta_{ij})}{\sum_j f(\Delta_{ij})}. \quad (3)$$

Add Edge. As before, no edge is added if the agent is already attached. An edge is added to an unattached agent using the product of Eq. (3) and the uniform agent selection probability, yielding $P_{E|G_t}\left[(a_i, s_j)|x_t\right] = \frac{P_{S|G_t}(s_j|x_t)}{|V_{\text{agents}}|}$

4.2 Detachment Probability

Edge $(a, s) \in E$ is first *selected* with uniform probability $1/|E|$. Second, the edge is *removed* using a detachment process from below.

Motivation for Detachment Processes. Biological models suggest that individual agents persist longer in states that favor higher quality sites.

- The length of a waggle dance and the number of assessment runs made by a honeybee is higher for high quality sites than low quality sites [41]
- Ants deposit more pheromone on returning from high quality foraging sites, creating trails that persist longer [12].

Quality-based detachment processes can exhibit individual preferences.

- Ants can have different grain size preference when building a nest [1].
- Birds can have different sugar preferences for nectar [29].

Baseline Detachment. An edge (a, s) randomly selected from the edge set is removed from the edge set with fixed probability 1/20. This value was subjectively set so that edges are probably not removed immediately after creation.

Nominal Detachment. Recall that $\text{qual}(s) \in [0, 1]$. The probability of removing (a, s) decreases with site quality, $P_{R|G_t}\left[(a, s)|x_t\right] = \left(1 - \text{qual}(s)\right)/|E|$.

Detachment with Quality Preferences. Individual agents may have preferences for different site qualities. Thus, we differentiate between the *objective* quality of a site, $\text{qual}(s_j)$ and the variations in the *subjective* site quality q_i^*, for agent a_i.The difference between the objective and subjective site qualities, $\delta_{ij} = \text{qual}(s_j) - q_i^*$, abstracts the differences in how agents assess site quality.

Homogeneous detachment experiments, where $\forall i \; q_i^* = q^* = 1$, are omitted because they are equivalent to nominal attachment.

Heterogeneous preferences mean that individual agents subjectively prefer sites of different qualities. The edge removal probability is defined as

$$P_{R|G_t}\left[(a_i, s_j)|x_t\right] = \begin{cases} \frac{1-\delta_{ij}}{|E|} & \text{if } \delta_{ij} \geq 0 \\ 0.25 & \text{otherwise} \end{cases} . \tag{4}$$

The probability of removing an edge is minimum when the objective quality is close to the subjective quality. When objective quality is less than the subjective preference, Eq. (4) assigns a uniform probability of detachment. Parameters for heterogeneous quality preferences are given in Section 5.

5 Methods

Twenty Agents and Five Sites. Pilot experiments indicated that 20 agents and 5 sites reasonably represent a range of larger colonies.

Twenty Trials and Runs. A single run consists of using one set of parameters from the cases in Table 1 and running the graph based simulation for 700 time steps. A trial is 20 runs with the same parameters. An experiment (or each case) consists of 20 trials with the same parameters. Multiple trials generate mean and interquartile range estimates for success probability. The initial graph configuration had no agent-site connections.

Quality Distributions. The probability of converging to a successful configuration depends on the objective site qualities. Linear, exponential, and sublinear distributions of quality represent many resource allocation problems [15], problems finding the most influential nodes in a social network [30], and problems like foraging [43], respectively. These distributions are subjectively chosen as $\mathbf{q}_{\text{lin}} = \theta$, $\mathbf{q}_{\text{exp}} = e^{5(\theta-1)}$, and $\mathbf{q}_{\text{sub}} = (\theta^{1/2})/2$, respectively, where $\theta = [0.2, 0.3, 0.5, 0.75, 0.9]$ is a parameter vector.

Experiment Conditions. Five attachment conditions (random, nominal, homogeneous, heterogeneous, and saturation), three detachment conditions (random, nominal, and heterogeneous), and three site quality distributions (linear, exponential, sublinear) were used. Note that random detachment ignores site quality. All combinations were considered, but results are shown only for interesting conditions. For homogeneous attachment $d^* = \frac{|V_{\text{agents}}|}{2} = 10$, for heterogeneous attachment the d_i^*'s were independently sampled from $\mathcal{N}(10, 2)$, for saturation $d^{sat} = 10$, and for heterogeneous detachment the q_i^*'s were independently sampled from $\mathcal{N}(1, 0.05)$. The same set of d_i^*, q_i^* and d^{sat} were used for each sample run within each trial set.

6 Results

Results are shown in Table 1. The first column is a reference number denoted *case*. Lines between rows and the daggers indicate comparison groups. The second column specifies the attachment, detachment, and quality conditions. The third column shows the sample mean and interquartile range for the probability of finding a successful configuration as a function of time. The plurality rule in these figures breaks ties in favor of the best site. A *successful configuration* is one in which more agents are connected to the site with highest objective quality than any other site. In all the figures, the brief initial high success rate is an artifact of ranking sites when no agents are attached.

The fourth column (Ratio of Sites) shows a stacked bar graph that indicates how often each site had the highest number of agents attached to it at the end of each set and trial. The top bar indicates the site with highest objective quality, the second bar indicates the site with second highest quality, and so on. The width of the bar is the percentage of time that site has the largest number of agents attached at the end of each simulation run. When there are multiple sites that tie for the most numbers of edges, the tie is broken in favor of the lowest quality site; this is done to present a conservative representation of ranking that balances the tie-breaker used in the average success plots. Case 9 shows two bars which show the percentage of time sites are ranked first (left stack) or second (right stack). The key to the stacked bar graphs is given in the first row. Objective site qualities are ranked $q(s_4) > q(s_3) > \ldots > q(s_0)$.

The fifth column shows typical graph configuration at the end of a simulation run, when showing the example is helpful. The monochromatic vertices in the upper left represent agents, and the colormapped vertices represent sites.

Baseline: Cases 1-2. Random attachment and detachment results (not shown in the table) are typified by the following: Each site is equally likely to be chosen as the best solution which means that the highest quality site is chosen about 1/5 of the time, the site chosen as best changes often over time, and a typical configuration has very few edges. Nominal attachment and random detachment in Case 1 show a similar pattern, where each site is equally likely to be chosen as the best-of-N solution and the highest quality site is chosen 1/5 of the time. Unlike random attachment/detachment, the site that is chosen under nominal attachment rarely changes after the first 100 time steps. The example configuration shows that popularity-based attachment causes many agents to cluster around a common site regardless of the site's quality.

Random attachment and nominal detachment in Case 2 show the following: the highest quality site is chosen more than 50% of the time because agents persist at the highest quality site longer than at other sites. However, clusters are small, as illustrated by the example configuration, making it difficult for the best site to "hold onto" agents for a long time. Linear and sublinear quality distributions decrease the success probability compared to the exponential quality distribution because there is less difference between site qualities.

Effect of Quality: Cases 3-5. Nominal (popularity-based) attachment and nominal (quality-based) detachment exhibit the following: The first or second

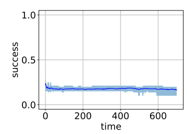
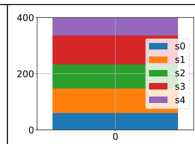
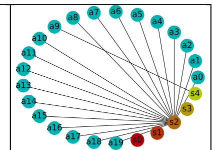
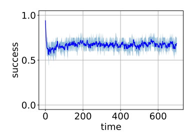
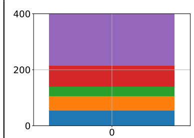
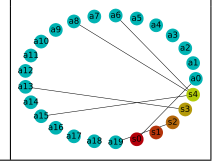
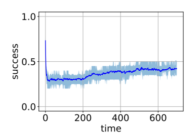
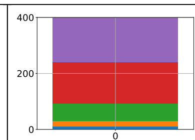
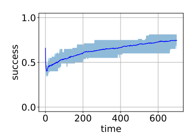
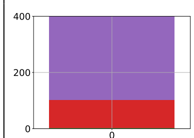
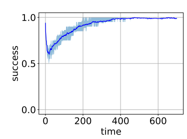
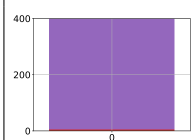
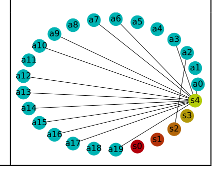
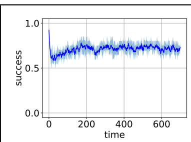
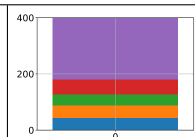
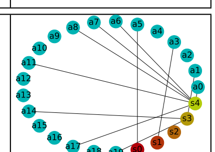
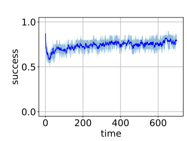
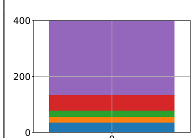
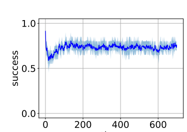
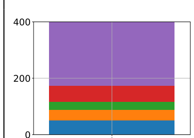
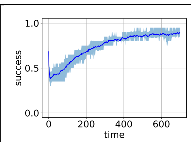
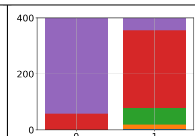
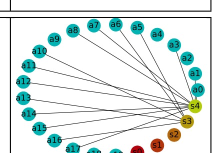
Case	Att, Det, Qual	Success	Ratio of sites	Example
1	Nom, Rand, N/A			
2	Rand, Nom, Exp			
3	Nom, Nom, Sub			
4	Nom, Nom, Lin			
5†	Nom, Nom, Exp			
6†	Hom, Nom, Exp			
7†	Het, Nom, Exp			
8†	Het, Het, Exp			
9	Sat, Nom, Lin			

Table 1: Results for different attachment (Att), detachment (Det), and quality distributions (Qual). Success probability over time, ratio of sites ranked as first, and example final graph configurations are shown.

highest quality sites are most likely chosen as the best-of-N solution, and the chosen site rarely changes after 100 iterations. The typical configuration shows that many agents cluster at the highest quality site. Comparing Cases 3-5 reveals that the type of quality distribution affects success. When the distance between site qualities is low (linear/sublinear), quality-based persistence is less effective, making other sites more likely to be chosen. Importantly, the red band in the stacked bar chart indicates that the second highest quality site is often chosen when two sites have nearly equal quality.

Effect of Heterogeneity: Cases $5^{\dagger}$-$8^{\dagger}$. Comparing Case 5, which has nominal attachment, to Case 6, which has homogeneous attachment, appears to show that homogeneity decreases the success probability. It is not homogeneity per se that is responsible for the differences in these cases since all agents in the nominal attachment condition behave the same. Rather, the difference is that the probability of attachment under homogeneous attachment increases as degree increases until it reached $d^* = |V_{\text{agents}}|/2 = 10$, but when more than 10 agents are attached to a site the probability of a new agent attaching to it decreases. By contrast, the nominal attachment probability grows linearly with degree. This results in lower average success rates under homogeneous attachment because it is easier for the probabilistic dynamics to induce switches between configurations that are constrained in their popularity.

Case 7 shows that heterogeneous agents are more likely than homogeneous agents to select the highest quality site. This is because there is a chance that agents will prefer sites with degree $d_i^* > |V_{\text{agents}}|/2 = 10$. This allows more agents to be attracted to more popular sites provided that agents who prefer less popular sites attach first and then stay attached due to the persistence of high quality sites. Case 8, which uses both heterogeneous attachment and detachment, shows that the highest quality site is selected less frequently than Case 7 (note the larger blue, orange, and green bands in the bar graph). Agents that subjectively prefer sites with lower objective quality persist longer at those sites while popularity-based attachment recruits other agents to the site.

Best-M-of-N: Cases 3,4,8,9. Case 9 indicates that saturating the effect of popularity causes agents to attach to other sites. As illustrated in the example configuration and the stacked bar chart, agents are likely to form a second cluster around the second highest quality site. In effect, popularity-based clustering is divided across multiple possible sites, allowing quality-based persistence to divide agents among the two best sites. Indeed, two stacked bar charts indicate that the two objectively highest quality sites are almost always selected as the best-two-of-N solutions. Note that an exponential distribution decreases the probability of selecting the second highest quality site because relative persistence at that site is lower, and a sublinear distribution increases the relative persistence at the second highest quality site. Cases 3,4,8,9 all exhibit a division of agent clusters around multiple sites. These suggest that algorithms that successfully solve the best-M-of-N will perform best when the algorithm includes some form of saturation, degree preference, and heterogeneity. Moreover, best-M-of-N might be easier to solve when when distances between site qualities are small.

7 Conclusion and Future Work

A graph-based abstraction of a hub-based colony can plausibly be used to solve the best-of-N problem. Moreover, degree-based preferential attachment combined with quality-based detachment appear sufficient for solving the problem. A colony is more likely to successfully solve the best-M-of-N problem when popularity-based saturation and agent heterogeneity are added to the algorithm.

Future work includes (a) modeling how configurations will evolve in real-time, (b) exploring the effects of noisy estimates of popularity and quality on the likelihood of finding the best solution, (c) finding useful blends of quality, popularity, and other environment information (e.g., site distance) in both attachment and detachment processes, (d) modeling existing agent-based models so that abstract analyses of these models can be performed, and (e) developing formal analysis tools for the graph random process to establish theoretical colony properties.

8 Acknowledgements

This paper was partially supported by a grant from the US Office of Naval Research under grant number N00014-18-1-2503. All opinions, findings, and results are the responsibility of the authors and not the sponsoring organization.

References

1. Aleksiev, A.S., Longdon, B., Christmas, M.J., Sendova-Franks, A.B., Franks, N.R.: Individual choice of building material for nest construction by worker ants and the collective outcome for their colony. Animal Behaviour **74**(3), 559–566 (2007)
2. Barabási, A.L., Albert, R.: Emergence of scaling in random networks. Science **286**(5439), 509–512 (1999)
3. Binitha, S., Sathya, S.S., et al.: A survey of bio inspired optimization algorithms. International journal of soft computing and engineering **2**(2), 137–151 (2012)
4. Brambilla, M., Ferrante, E., Birattari, M., Dorigo, M.: Swarm robotics: a review from the swarm engineering perspective. Swarm Intelligence **7**(1), 1–41 (2013)
5. Brown, J.L.: Optimal group size in territorial animals. Journal of Theoretical Biology **95**(4), 793–810 (1982)
6. Bussemaker, H.J., Deutsch, A., Geigant, E.: Mean-field analysis of a dynamical phase transition in a cellular automaton model for collective motion. Physical Review Letters **78**(26), 5018 (1997)
7. Carrillo, J.A., Choi, Y.P., Hauray, M.: The derivation of swarming models: mean-field limit and wasserstein distances. In: Collective dynamics from bacteria to crowds, pp. 1–46. Springer (2014)
8. Chow, Y., Moriguti, S., Robbins, H., Samuels, S.: Optimal selection based on relative rank (the "secretary problem"). Israel Journal of Mathematics **2**(2), 81–90 (1964)
9. Cody, J.R., Adams, J.A.: An evaluation of quorum sensing mechanisms in collective value-sensitive site selection. In: 2017 International Symposium on Multi-Robot and Multi-Agent Systems (MRS). pp. 40–47. IEEE, Los Angeles, CA (Dec 2017)

10. Coppola, M., Guo, J., Gill, E., De Croon, G.C.: The pagerank algorithm as a method to optimize swarm behavior through local analysis. Swarm Intelligence **13**(3-4), 277–319 (2019)
11. Dorigo, M., Birattari, M., Blum, C., Christensen, A.L., Reina, A., Trianni, V.: Ant colony optimization and swarm intelligence: 11th international workshop, ANTS 2018. proceedings. Lecture Notes in Computer Science **LNCS-11172** (2018)
12. Dorigo, M., Bonabeau, E., Theraulaz, G.: Ant algorithms and stigmergy. Future Generation Computer Systems **16**(8), 851–871 (2000)
13. Estes, R.D.: Social organization of the african bovidae. The behaviour of ungulates and its relation to management **1**, 166–205 (1974)
14. Franks, N.R., Dornhaus, A., Best, C.S., Jones, E.L.: Decision making by small and large house-hunting ant colonies: one size fits all. Animal behaviour **72**(3), 611–616 (2006)
15. Fu, T.p., Liu, Y.s., Chen, J.h.: Improved genetic and ant colony optimization algorithm for regional air defense wta problem. In: First International Conference on Innovative Computing, Information and Control-Volume I (ICICIC'06). vol. 1, pp. 226–229. IEEE (2006)
16. Gao, L., Wang, Y., Tang, Z., Lin, X.: Newspaper article reconstruction using ant colony optimization and bipartite graph. Applied Soft Computing **13**(6), 3033–3046 (2013)
17. Gerard, J.F., Bideau, E., Maublanc, M.L., Loisel, P., Marchal, C.: Herd size in large herbivores: encoded in the individual or emergent? The Biological Bulletin **202**(3), 275–282 (2002)
18. Ghaffari, M., Musco, C., Radeva, T., Lynch, N.: Distributed house-hunting in ant colonies. In: Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing. pp. 57–66. ACM (2015)
19. Goodrich, M., Jain, P.: Swarm Intelligence: 12th International Conference, ANTS 2020, Extended Abstracts. Springer (2020)
20. Gordon, D.M.: Ant encounters: interaction networks and colony behavior, vol. 1. Princeton University Press (2010)
21. Hoare, D.J., Couzin, I.D., Godin, J.G., Krause, J.: Context-dependent group size choice in fish. Animal Behaviour **67**(1), 155–164 (2004)
22. Imirzian, N., Zhang, Y., Kurze, C., Loreto, R.G., Chen, D.Z., Hughes, D.P.: Automated tracking and analysis of ant trajectories shows variation in forager exploration. Scientific reports **9**(1), 1–10 (2019)
23. Jakob, E.M.: Individual decisions and group dynamics: why pholcid spiders join and leave groups. Animal Behaviour **68**(1), 9–20 (2004)
24. Kempe, D., Kleinberg, J., Tardos, É.: Maximizing the spread of influence through a social network. In: Proceedings of the ninth ACM SIGKDD International Conference on Knowledge Discovery and Data mining. pp. 137–146 (2003)
25. Kimura, M., Saito, K., Nakano, R., Motoda, H.: Extracting influential nodes on a social network for information diffusion. Data Mining and Knowledge Discovery **20**(1), 70–97 (Jan 2010)
26. Laomettachit, T., Termsaithong, T., Sae-Tang, A., Duangphakdee, O.: Decision-making in honeybee swarms based on quality and distance information of candidate nest sites. Journal of theoretical biology **364**, 21–30 (2015)
27. Lee, C., Lawry, J., Winfield, A.: Negative updating combined with opinion pooling in the best-of-N problem in swarm robotics. In: International Conference on Swarm Intelligence. pp. 97–108. Springer (2018)
28. Leonard, N.E.: Multi-agent system dynamics: Bifurcation and behavior of animal groups. Annual Reviews in Control **38**(2), 171–183 (2014)

29. Lotz, C.N., Schondube, J.E.: Sugar preferences in nectar-and fruit-eating birds: Behavioral patterns and physiological causes 1. Biotropica: The Journal of Biology and Conservation **38**(1), 3–15 (2006)
30. Lusher, D., Koskinen, J., Robins, G.: Exponential random graph models for social networks: Theory, methods, and applications. Cambridge University Press (2013)
31. Mesbahi, M., Egerstedt, M.: Graph theoretic methods in multiagent networks. Princeton University Press (2010)
32. Moffett, M.W.: The human swarm: how our societies arise, thrive, and fall. Basic Books (2019)
33. Nedić, A., Olshevsky, A., Uribe, C.A.: Nonasymptotic convergence rates for cooperative learning over time-varying directed graphs. In: 2015 American Control Conference (ACC). pp. 5884–5889. IEEE (2015)
34. Nevai, A.L., Passino, K.M., Srinivasan, P.: Stability of choice in the honey bee nest-site selection process. Journal of Theoretical Biology **263**(1), 93–107 (2010)
35. Newman, M.E.: Clustering and preferential attachment in growing networks. Physical review E **64**(2), 025102 (2001)
36. Newman, M.E.: Properties of highly clustered networks. Physical Review E **68**(2), 026121 (2003)
37. Omic, J., Kooij, R., Van Mieghem, P.: Virus spread in complete bi-partite graphs. In: 2nd International ICST Conference on Bio-Inspired Models of Network, Information, and Computing Systems (2008)
38. Passino, K.M.: Biomimicry of bacterial foraging for distributed optimization and control. IEEE control systems magazine **22**(3), 52–67 (2002)
39. Prasetyo, J., De Masi, G., Ranjan, P., Ferrante, E.: The best-of-N problem with dynamic site qualities: Achieving adaptability with stubborn individuals. In: International Conference on Swarm Intelligence. pp. 239–251. Springer (2018)
40. Ren, W., Beard, R.W., Atkins, E.M.: A survey of consensus problems in multiagent coordination. In: Proceedings of the 2005, American Control Conference, 2005. pp. 1859–1864. IEEE (2005)
41. Seeley, T.D., Buhrman, S.C.: Nest-site selection in honey bees: how well do swarms implement the "best-of-N" decision rule? Behavioral Ecology and Sociobiology **49**(5), 416–427 (2001)
42. Shyu, S.J., Lin, B.M., Hsiao, T.S.: Ant colony optimization for the cell assignment problem in pcs networks. Computers & Operations Research **33**(6), 1713–1740 (2006)
43. Sinervo, B.: Optimal foraging theory: constraints and cognitive processes. University of Southern California Santa Cruz: available at printfu. org/foraging+ animals pp. 105–130 (1997)
44. Sumpter, D.J.: Collective animal behavior. Princeton University Press (2010)
45. Sumpter, D.J., Beekman, M.: From nonlinearity to optimality: pheromone trail foraging by ants. Animal behaviour **66**(2), 273–280 (2003)
46. Valentini, G., Ferrante, E., Dorigo, M.: The best-of-N problem in robot swarms: Formalization, state of the art, and novel perspectives. Frontiers in Robotics and AI **4** (Mar 2017)
47. Wilson, J.G.: Optimal choice and assignment of the best m of n randomly arriving items. Stochastic processes and their applications **39**(2), 325–343 (1991)
48. Zhang, Y., Zhou, J., Cheng, J.: Preference-based top-K influential nodes mining in social networks. In: 2011 IEEE 10th International Conference on Trust, Security and Privacy in Computing and Communications. pp. 1512–1518. IEEE, Changsha, China (Nov 2011)