A Generalized Method to Find the Square Root of Matrix Whose Characteristic Equation Is Quadratic

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Abstract

In this paper, we generalized the method to calculating the square root of matrix whose characteristic is quadratic and how to Cayley-Hamilton theorem may be used to determine the formula for all square root of matrix whose order is 2×2.

Keywords : Eigen values, Matrix equation, Square root of matrix.

1. Introduction

Let $M_n(C)$ be the set of all complex matrices whose order is n×n. Matrix Q is said to be a square root of matrix P, if the matrix product Q.Q = P.

Now, what is the square root of matrix such as $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$. It is not, in general $\begin{bmatrix} \sqrt{p} & \sqrt{q} \\ \sqrt{r} & \sqrt{s} \end{bmatrix}$.

It is easy to see that the upper left entry of its square is $p + \sqrt{q}$ and not p.

In recent years, several article have been written about the root of a matrix, and one can refer to [4-6]. A number of method have been proposed to computing the square root of matrix and these are usually based on Newton's method, either directly or the sign function(see e.g., [1-3]).

2. Generalized Method

The set of all matrices which their square is P, denoted by $\sqrt{P}$, i.e.,

$$\sqrt{P} = \{Y\colon Y \in M_n(C), Y^2 = P\}$$

This set can be very large .For example, we will see that $\sqrt{I}$ has infinite members. We can define the n-th root of a matrix P as follows.

$$\sqrt[n]{P} = \{Y\colon Y \in M_n(C), Y^n = P\}$$

It is well known to all, if P = $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, then characteristic equation is

$$\lambda^2 - (Trace\ P)\lambda + detP = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Apply cayley - Hamilton theorem, putting $\lambda = P$, then equation (1) is

$$P^2 - (Trace\ P)P + (detP)I = 0$$

Thus, we have $P^2 = (Trace\ P)P - (detP)I \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

Putting , $P^2 = Q$, then equation (2) is

$$Q = (Trace\ P)P - (detP)I$$
$$Q + (detP)I = (Trace\ P)P$$
$$\frac{1}{(Trace\ P)}[Q + (detP)I] = P \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Lemma 2.1. Let P be a 2×2 matrix. Then $trace\ P^2 = (traceP)^2 - 2\ detP$.

Proof : Suppose λ_1 and λ_2 are the two eigen values of the matrix P. Then we can easy to see that λ_1^2 and λ_2^2 are the eigen values of P^2. We know that , $trace\ P = \lambda_1 + \lambda_2$ and $detP = \lambda_1\lambda_2$.

Then, $trace\ P^2 = {\lambda_1}^2 + {\lambda_2}^2$

$$= (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2$$
$$= (trace\ P)^2 - 2\det P$$

Second proof: In other words, let P = $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Then,

$$P^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$
$$P^2 = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

Therefore,

$$Trace\ P^2 = (p^2 + rq) + (s^2 + rq)$$
$$Trace\ P^2 = p^2 + s^2 + 2rq$$
$$Trace\ P^2 = p^2 + s^2 + 2ps - 2ps + 2rq$$
$$Trace\ P^2 = (p + s)^2 - 2(ps - rq) \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

But, $trace\ P = p + s$ and $\det P = ps - qr$, then equation (1),

$$Trace\ P\text{Let P, Q} \in \mathrm{M_n(C)}\ ^2 = (trace\ P)^2 - 2\det P$$

Remark.1. Let P, Q ∈ $M_2(C)$ and $P^2 = Q$. Then the following statements are holds:

(1) $\det P = \sqrt{\det Q}$

(2) $\operatorname{tracet} P = \sqrt{\operatorname{trace} Q + 2\sqrt{\det Q}}$

Example 2.1 Let $Q = \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix}$. So detQ = 64-15 = 49, and traceQ = 8+8 = 16, therefore if $P^2 = Q$,then,

$$\det P = \sqrt{\det Q} = \sqrt{49} = \pm 7, \text{ and}$$

$trace\ P = \sqrt{\operatorname{trace} Q + 2\sqrt{\det Q}} = \sqrt{16 + 2\sqrt{49}} = \sqrt{16 \pm 14}$, taking positive and negative sign then ,

$trace\ P = \pm\sqrt{30}\ or\ trace\ P = \pm\sqrt{2}$, thus, from equation (3),

$$P = \frac{1}{(Trace\ P)}[Q + (detP)I],$$

$$P = \frac{1}{\pm\sqrt{30}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (\pm 7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ or } P = \frac{1}{\pm\sqrt{2}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (\pm 7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Therefore,

$$P = \frac{1}{\pm\sqrt{30}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ or } P = \frac{1}{\pm\sqrt{30}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ and}$$

$$P = \frac{1}{\pm\sqrt{2}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ or } P = \frac{1}{\pm\sqrt{2}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, \text{ on calculating then we have,}$$

$$P = \pm\frac{1}{\sqrt{30}}\begin{bmatrix} 15 & 5 \\ 3 & 15 \end{bmatrix} \text{ or } P = \pm\frac{1}{\sqrt{30}}\begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}, \text{ and } P = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 15 & 5 \\ 3 & 15 \end{bmatrix} \text{ or } P = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}$$

Lemma 2.2 Let P ∈ $M_2(C)$. If trace P = 0, then $P^2 \in \langle I \rangle$.

Proof. We will prove this lemma in two ways. In general, we have

$$P^2 - (Trace\ P)P + (detP)I = 0 \ldots\ldots\ldots\ldots(1)$$

Therefore, if trace P = 0, then from (1) we obtain,

$$P^2 + (detP)I = 0$$

$$P^2 = -(detP)I \text{ and } P^2 \in \langle I \rangle$$

Second proof: , let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, and $p + s = 0$

Then,

$$P^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$P^2 = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

Putting $p = -s$, then

$$P^2 = \begin{bmatrix} p^2 + rq & 0 \\ 0 & s^2 + rq \end{bmatrix}$$

Hence , when $p^2 = s^2$ then $P^2 = (p^2 + rq)$.

Example 2.2 Let $Q = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Then $detQ = 2 - 6 = -4$, and $trace\ Q = 1 + 2 = 3$. If $P^2 = Q$ then

$$\det P = \sqrt{\det Q} = \sqrt{-4} = 2i, \text{ and}$$

$$trace\ P = \sqrt{\operatorname{trace} Q + 2\sqrt{\det Q}}$$

$$= \sqrt{3 + 2\sqrt{-4}}$$

$$= \sqrt{3 + 4i},$$

Now,

$$P = \frac{1}{(Trace\ P)}[Q + (detP)I],$$

$$P = \frac{1}{\pm\sqrt{3+4i}}\left\{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} + 2i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$P = \frac{1}{\pm\sqrt{3+4i}}\left\{\begin{bmatrix} 1 + 2i & 3 \\ 2 & 2 + 2i \end{bmatrix}\right\}$$

Lemma 2.3 For each $\beta \in C$ and any matrix P, $\sqrt{\beta P} = \sqrt{\beta}\sqrt{P}$.

Proof: Suppose that $\beta \neq 0$ and $Y \in \sqrt{BP}$. So $Y^2 \in \beta P$, hence $\frac{Y}{\sqrt{\beta}} \in \sqrt{P}$, which implies that $Y \in \sqrt{\beta}\sqrt{P}$.

Conversely, if $Y \in \sqrt{BP}$, then $\frac{Y^2}{\beta} = P$. Hence $Y^2 = \beta P$ and $Y \in \sqrt{\beta P}$.

Now, we try to compute $\sqrt{I}$. Suppose that P ∈ $M_2(C)$ and $P^2 = I$. Let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

Then,

$$P^2 = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}, \text{ but } P^2 = I, \text{ then}$$

$$I = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

Hence we have,

$$p^2 + rq = 1 \ldots\ldots\ldots\ldots(1)$$

$$pq + qs = 0 \ldots\ldots\ldots\ldots(2)$$

$$pr + rs = 0 \ldots\ldots\ldots\ldots(3)$$

$$s^2 + rq = 1 \ldots\ldots\ldots\ldots(4)$$

From (2) and (3), q = 0 or p + s = 0 and r = 0 or p + s =0. We consider two cases:

(1) If p + s = 0, then equation (2) and (3) hold. We have $p^2 + rq = 1$or $p = \sqrt{1-rq}$ and since a + d = 0 and since p + s = 0 we have p = -s $= -\sqrt{1-rq}$. Therefore

$$P = \left\{ \begin{bmatrix} \sqrt{1-rq} & 0 \\ 0 & -\sqrt{1-rq} \end{bmatrix} : b, c \in C \right\}.$$

(2) If $p + s \neq 0$ we must have q = 0 and r = 0. Hence $p = \pm 1$and $s = \pm 1$. Therefore there are two solutions $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Hence we can write

$$\sqrt{I} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cup \begin{bmatrix} \sqrt{1-rq} & 0 \\ 0 & -\sqrt{1-rq} \end{bmatrix} b, c \in C \right\}.$$

Example 2.3 Let $Q = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$. Therefore $Q = 16 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 16$ I. Then $\sqrt{Q} = 4\sqrt{I}$, hence we have

$$\sqrt{I} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \cup \begin{bmatrix} 4\sqrt{1-rq} & 2q \\ 2r & -4\sqrt{1-rq} \end{bmatrix} : b, c \in C \right\}$$

References

[1] G. Alefeld, N. Schneider, On square roots of M-matrices, Linear. Algebra. Appl. 42 (1982), 119-132.
[2] E. D. Denman, A. N. Beavers, The matrix sign function and computations in systems, Appl. Math. Comput. 2 (1) (1976), 63-94.
[3] W. D. Hoskins, D. J. Walton, A faster method of computing the square root of a matrix, IEEE Trans. Automat. Control. 23 (3) (1978), 494-495.
[4] B. W. Levinger, The square root of a 2×2 matrix, Math. Mag. 53 (4) (1980), 222-224.
[5] A. Nazari, H. Fereydooni, M. Bayat, A manual approach for calculating the root of square matrix of dimension, Math. Sci. 7 (1) (2013), 1-6.
[6] D. Sullivan, The square roots of 2 × 2 matrices, Math. Mag. 66 (5) (1993), 314-316.