Diffusion and Brownian Motion Processes in Modeling the Costs of Supporting Non-Autonomous Pension Funds

Manuel Alberto M. Ferreira and José António Filipe

May 8, 2022

Diffusion and Brownian motion processes in modeling the costs of supporting non-autonomous pension funds

M. A. M. Ferreira[1] and J. A. Filipe[2]

[1]University Institute of Lisbon (ISCTE – IUL), Information Sciences, Technologies, and Architecture Research Center (ISTAR-IUL), Business Research Unit (BRU-IUL), Lisboa 1649-026, Portugal
[2]Department of Mathematics, ISTA— School of Technology and Architecture, University Institute of Lisbon (ISCTE-IUL), Information Sciences, Technologies, and Architecture Research Center (ISTAR-IUL), Business Research Unit (BRU-IUL), Lisboa 1649-026, Portugal

Abstract

In this chapter, we consider pensions funds not sufficiently auto financed and systematically maintained with an outside financing effort, usually non-autonomous pension's funds. This financial effort, made by the managing entity, translates as capital injections into the fund. The objective of this work is to develop a tool that allows predicting the appropriate moments to carry out these interventions and the respective amounts. So, we propose to represent the unrestricted reserves value process of this kind of funds, through a time homogeneous diffusion process with finite expected time till the ruin. A financial tool that regenerates the diffusion is also admitted, at some level with positive value every time it hits a barrier at the origin. Then the financing effort may be modeled as a renewal-reward process if the regeneration level is kept constant. The perpetual maintenance cost expected values evaluation and of the finite time maintenance cost are studied. Then, we focus on a particular situation of this approach, arising when the unrestricted reserves value process behaves as a generalized Brownian motion process.

Keywords: Pensions fund, diffusion process, first passage times, renewal equation.

1 Introduction

Pension funds represent savings collected along people's working life. Pension funds that support personal pension plans are intended to be autonomous. They represent the highest level of protection to the beneficiary from the bankruptcy of the sponsor, especially when the custodian is involved. Non-autonomous pension funds are not legally separated from the plan sponsor but are kept on its balance sheet. In this case, there is the lowest protection level to the beneficiary from bankruptcy of the sponsor, since the sponsor can use pension's assets to fund its business, see Impavido (2012).

The financial problem of asset-liability management scheme of a pensions fund requires a management program that demands a set of decisions. In particular, the amounts and the instants at which it is necessary to inject money in the fund in order to keep it sustainable. Sponsors are obviously interested in an appropriate management of the risk for their pension funds. Well and balanced funded pension funds result essential in this process of funds management.

Through this chapter we develop a mathematical tool that allows predicting, in a probabilistic mode, the appropriate moments to carry out these money injections and the respective amounts.

This issue is particularly relevant since we know that pension funds are continuously exposed to the market's situation. And the recent financial crises and turbulent stock markets circumstances made the problem of pension's funds management to receive an enormous attention. Many pensions' funds suffered dramatic losses, and this is a problematic issue that managers want to overcome the best they can. So, managerial tools allow a better decision-making.

The protection cost present value expectation for a non-autonomous pensions' fund is considered in this work. Two contexts are considered:

- The protection effort is perpetual,
- The protection effort happens for a finite time period.

It is admitted that the unrestricted fund reserves behavior may be modeled as a time homogeneous diffusion process. Then a regeneration scheme of the diffusion to include the effect of an external financing effort is used.

This chapter is an updated and enlarged version of Ferreira (2012), where was mainly considered the diffusion process.

In Gerber and Parfumi (1998) a similar work is presented. A Brownian motion process conditioned by a reflection scheme was considered. With less constraints, but in different conditions, exact solutions were then obtained for both problems.

The work presented in Refait (2000), on asset-liability management aspects, also motivated the use of the Brownian motion application example in that domain.

So, in this chapter we extend the results presented in Ferreira (2012), better specifying the diffusion process mathematical details, and deeply exploring the Brownian motion process situation

Other works on this subject are Figueira and Ferreira (2003) and Figueira (2003), both dealing with the diffusion process case. The works Filipe, Ferreira and Andrade (2012), Andrade et al. (2012), Andrade et al. (2012), Ferreira et al. (2011) and Ferreira et al. (2012) deal with other financial problems, slightly different from the presently considered here, but relevant to their understanding and framing. In particular Andrade et al. (2012) and Ferreira et al. (2012) present the problem of state pension funds, in which workers 'contributions are currently insufficient to pay pensioners' pensions due to demographic imbalances that occur in modern societies. In this case, state budgets have to include capitals to balance these funds. The tool that we are going to develop can be applied in this situation, contributing for the transfers to be made in a scheduled manner, at the times and amounts due, in a more efficient way.

2 Pensions Fund Reserves Behavior Stochastic Model

Be $X(t), t \geq 0$ the reserves value process of a pensions fund given by an initial reserve amount $a, a > 0$ added to the difference between the total amount of contributions received and the total amount of pensions paid both up to time t. It is assumed that $X(t)$ is a time homogeneous diffusion process, with $X(0) = a$, defined by drift and diffusion coefficients $\mu(x)$ and $\sigma^2(x)$, respectively.

Call S_a the first passage time of $X(t)$ by 0, coming from a. The funds to be considered in this work are non-autonomous funds. So

$$E[S_a] < \infty, \text{for any } a > 0 \qquad (2.1).$$

That is: funds where the pensions paid consume in finite expected time any initial positive reserve and the contributions received. Then other financing resources are needed in order that the fund survives.

The condition (2.1) may be fulfilled for a specific diffusion process using criteria based on the drift and diffusion coefficients. In this context, here the work presented in Bhattacharya and Waymire (1990), pg. 418-422, is followed. So, accept that $P(S_a < \infty) = 1$ if the diffusion scale function is $q(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{z}\frac{2\mu(y)}{\sigma^2(y)}dy} dz$, where x_0 is a diffusion state space fixed arbitrary point, fulfilling $q(\infty) = \infty$.

Then the condition (2.1) is equivalent to $p(\infty) < \infty$, where $p(x) = \int_{x_0}^{x} \frac{2}{\sigma^2(z)} e^{\int_{x_0}^{z}\frac{2\mu(y)}{\sigma^2(y)}dy} dz$, be the diffusion speed function.

It is admitted that whenever the exhaustion of the reserves happens an external source place instantaneously an amount $\theta, \theta > 0$ of money in the fund so that it may keep on performing its function.

The reserves value process conditioned by this financing scheme is denoted by the modification $\breve{X}(t)$ of $X(t)$ that restarts at the level θ whenever it hits 0. As $X(t)$ was defined as a time homogeneous diffusion, $\breve{X}(t)$ is a regenerative process. Call $T_1, T_2, T_3, \ldots$ the sequence of random variables where T_n denotes the n^{th} $\breve{X}(t)$ passage time by 0. It is obvious that the sequence of time intervals between these hitting times $D_1 = T_1, D_2 = T_2 - T_1, D_3 = T_3 - T_2, \ldots$ is a sequence of independent random variables where D_1 has the same probability distribution as S_a and $D_2, D_3, \ldots$ the same probability distribution as S_θ.

3 First Passage Times Laplace Transforms

Call $f_a(s)$ the probability density function of $S_a(\text{related to } D_1)$. The corresponding probability distribution function is denoted by $F_a(s)$. The Laplace transform of S_a is denoted $\varphi_a(\lambda)$.

Consequently, the density, distribution and transform of S_θ $(\text{related to } D_2, D_3, \ldots)$ will be denoted by $f_\theta(s), F_\theta(s)$ and $\varphi_\theta(\lambda)$, respectively.

The transform $\varphi_a(\lambda)$ satisfies the second order differential equation

$$\begin{aligned} &\frac{1}{2}\sigma^2(a)u_\lambda''(a) + \mu(a)u_\lambda'(a) = \lambda u_\lambda(a), \\ &u_\lambda(a) = \varphi_a(\lambda), u_\lambda(0) = 1, u_\lambda(\infty) = 0, \end{aligned} \qquad (3.1).$$

See Feller (1971), pg. 478, Karlin and Taylor (1981), pg. 243 and Bass (1998), pg. 89.

4 Perpetual Maintenance Cost Present Value

Consider the perpetual maintenance cost present value of the pension's fund given by the random variable $V(r, a, \theta) = \sum_{n=1}^{\infty} \theta e^{-rT_n}$, r>0, where r represents the appropriate discount rate. Note that $V(r, a, \theta)$ is a random perpetuity. What matters is its expected value which is simple to calculate through Laplace transforms. Since the T_n Laplace transform is

$$E\left[e^{-\lambda T_n}\right] = \varphi_a(\lambda)\varphi_\theta^{n-1}(\lambda),$$

$$v_r(a, \theta) = E[V(r, a, \theta)] = \frac{\theta\varphi_a(r)}{1 - \varphi_\theta(r)} \qquad (4.1).$$

It is relevant to note that

$$\lim_{\theta \to 0} v_r(a, \theta) = \frac{u_r(a)}{-u_r'(0)} \qquad (4.2).$$

5 Finite Time Period Maintenance Cost Present Value

Define the renewal process $N(t)$ as $N(t) = \sup\{n: T_n \le t\}$, generated by the extended sequence $T_0 = 0, T_1, T_2, \ldots$. The present value of the pensions fund maintenance cost up to time t is represented by the stochastic process

$W(t; r, a, \theta) = \sum_{n=1}^{N(t)} \theta e^{-rT_n}$, $W(t; r, a, \theta) = 0$ if $N(t) = 0$.

To calculate the expected value function of the process evaluation: $w_r(t;a,\theta) = E[W(t;r,a,\theta)]$, begin noting that it may be expressed as a numerical series. Indeed, evaluating the expected value function conditioned by $N(t) = n$, it is obtained $E[W(t;r,a,\theta)|N(t) = n] = \theta\varphi_a(r)\frac{1-\varphi_\theta^n(r)}{1-\varphi_\theta(r)}$.

Repeating the expectation:

$$w_r(t;a,\theta) = E[E[W(t;r,a,\theta)]|N(t)] = \theta\varphi_a(r)\frac{1-\gamma(t,\varphi_\theta(r))}{1-\varphi_\theta(r)} \qquad (5.1).$$

Here $\gamma(t,\xi)$ is the probability generating function of $N(t)$.

Denote now the T_n probability distribution function by $G_n(s)$ and assume $G_0(s) = 1$, for $s \geq 0$. Recalling that $P(N(t) = n) = G_n(t) - G_{n+1}(t)$, the above mentioned probability generating function is

$$\gamma(t,\xi) = \sum_{n=0}^{\infty} \xi^n P(N(t) = n) = 1 - (1-\xi)\sum_{n=1}^{\infty} \xi^{n-1} G_n(t) \qquad (5.2).$$

Substituting (5.2) in (5.1), $w_r(t;a,\theta)$ is expressed in the form of the series

$$w_r(t;a,\theta) = \theta\varphi_a(r)\sum_{n=1}^{\infty} \varphi_\theta^{n-1}(r)\, G_n(t) \qquad (5.3).$$

Call the $w_r(t;a,\theta)$ ordinary Laplace transform $\psi(\lambda)$. The probability distribution function $G_n(s)$, of T_n, ordinary Laplace transform is given $\varphi_a(\lambda)\frac{\varphi_\theta^{n-1}(\lambda)}{\lambda}$ and performing the Laplace transforms in both sides of (5.3) it is obtained $\psi(\lambda) = \frac{\theta\varphi_a(r)\varphi_a(\lambda)}{\lambda(1-\varphi_\theta(r)\varphi_\theta(\lambda))}$ or

$$\psi(\lambda) = \theta\varphi_a(r)\frac{\varphi_a(\lambda)}{\lambda} + \psi(\lambda)\varphi_\theta(r)\varphi_\theta(\lambda) \qquad (5.4).$$

Inverting Laplace transforms in both sides of (5.4) the following defective renewal equation is got:

$$w_r(t;a,\theta) = \theta\varphi_a(r)F_a(t) + \int_0^t w_r(t-s;a,\theta)\varphi_\theta(r)f_\theta(s)ds \qquad (5.5).$$

Now an asymptotic approximation of $w_r(t;a,\theta)$ will be obtained through the key renewal theorem, see Bhattacharya and Waymire (1990), pg. 376.

If in (5.5) $t \to \infty$

$$w_r(\infty;a,\theta) = \theta\varphi_a(r) + w_r(\infty;a,\theta)\varphi_\theta(r) \qquad (5.6).$$

Or $w_r(\infty;a,\theta) = \frac{\theta\varphi_a(r)}{1-\varphi_\theta(r)} = v_r(a,\theta)$.

This is the expression (4.1) for $v_r(a,\theta)$. Subtracting each side of (5.6) from each side of (5.5), and performing some elementary calculations the following, still defective, renewal equation

$$J(t) = j(t) + \int_0^t J(t-s)\varphi_\theta(r)f_\theta(s)ds \quad (5.7).$$

Here $J(t) = w_r(\infty; a,\theta) - w_r(t; a,\theta)$ and $j(t) = \theta\varphi_a(r)\big(1 - F_a(t)\big) + \frac{\theta\varphi_a(r)\varphi_\theta(r)}{1-\varphi_\theta(r)}\big(1 - F_\theta(t)\big)$.

Now, to obtain a common renewal equation from (5.7), it must be admitted the existence of a value $k > 0$ such that $\int_0^\infty e^{ks}\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta(-k) = 1$.

So, the transform $\varphi_\theta(\lambda)$ is defined in a domain different from the initially considered. That is, a domain including a convenient subset of the negative real numbers.

Multiplying both sides of (5.7) by e^{kt} the common renewal equation desired is finally obtained: $e^{kt}J(t) = e^{kt}j(t) + \int_0^t e^{k(t-s)}J(t-s)e^{ks}\varphi_\theta(r)f_\theta(s)ds$ from which, by the application of the key renewal theorem, it results

$$\lim_{t\to\infty} e^{kt}J(t) = \frac{1}{k_0}\int_0^\infty e^{ks}j(s)\,ds \quad (5.8).$$

And $k_0 = \int_0^\infty se^{ks}\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta'(-k)$, since $e^{kt}j(t)$ is directly Riemann integrable. The integral in (5.8) may expressed in terms of transforms as $\int_0^\infty e^{ks}j(s)\,ds = \frac{\theta\varphi_a(r)\varphi_a(-k)}{k}$.

So, an asymptotic approximation, in the sense of (5.8) was obtained:

$$w_r(t; a,\theta) \approx v_r(a,\theta) - c_r(a,\theta)e^{-k_r(\theta)t} \quad (5.9).$$

Here $k_r(\theta)$ is the positive value of k that fulfills:

$$\varphi_\theta(r)\varphi_\theta(-k) = 1 \qquad (5.10).$$

And

$$c_r(a,\theta) = \frac{\theta\varphi_a(r)\varphi_a\big(-k_r(\theta)\big)}{-k_r(\theta)\varphi_\theta(r)\varphi_\theta'(-k_r(\theta))} \qquad (5.11).$$

6 Brownian motion Example

Suppose the diffusion process $X(t)$, underlying the reserves value behavior of the pension's fund, is a generalized Brownian motion process, with drift $\mu(x) = \mu, \mu <$

0 and diffusion coefficient $\sigma^2(x) = \sigma^2, \sigma > 0$. Observe that the setting satisfies the conditions that were assumed above in this work. Namely $\mu < 0$ implies condition (2.1). Everything else remaining as previously stated, it will be proceeded to present the consequences of this particularization. In general, it will be added $(*)$ to the notation used before because it is intended to use these specific results later.

To obtain the first passage time S_a Laplace transform, remember (3.1), it must be solved the equation: $\frac{1}{2}\sigma^2(a)u_\lambda^{*''}(a) + \mu(a)u_\lambda^{*'}(a) = \lambda u_\lambda^*(a), u_\lambda^*(a) = \varphi_a(\lambda), u_\lambda^*(0) = 1$ $u_\lambda^*(\infty) = 0$. This is a homogeneous second order differential equation with constant coefficients, which general solution is $u_\lambda^*(a) = \beta_1 e^{\alpha_1 a} + \beta_2 e^{\alpha_2 a}$, with $\alpha_1, \alpha_2 = \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}$.

Condition $u_\lambda^*(\infty) = 0$ implies $\beta_1 = 0$ and $u_\lambda^*(0) = 1$ implies $\beta_2 = 1$ so that the solution is achieved:

$$u_\lambda^*(a) = e^{-K_\lambda a}\left(= \varphi_a^*(\lambda)\right), K_\lambda = \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \qquad (6.1).$$

In this case, the perpetual maintenance cost present value of the pensions fund is given by, following (4.1) and using (6.1),

$$v_r^*(a, \theta) = \frac{\theta e^{-K_r a}}{1 - e^{-K_r \theta}} \qquad (6.2).$$

Note that $v_r^*(a, \theta)$ is a decreasing function of the first variable and an increasing function of the second. Proceeding as before, in particular:

$$\lim_{\theta \to 0} v_r^*(a, \theta) = \frac{e^{-K_r a}}{K_r} \qquad (6.3).$$

This expression has been obtained in Gerber and Parfumi (1998), in a different context and using different methods but, obviously, with identical significance. In Gerber and Parfumi (1998), the authors acted with a generalized Brownian motion, with no constraints in what concerns the drift coefficient, conditioned by a reflection scheme at the origin.

A way to reach an expression for the finite time period maintenance cost present value, is starting by the computation of $k_r^*(\theta)$, solving (5.10). This means to determine a positive number k satisfying $e^{-K_r \theta} e^{-K_{-\lambda}\theta} = 1$ or $K_r + K_{-\lambda} = 0$.

This identity is verified for the value of k:

$$k_r^*(\theta) = \frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \qquad (6.4).$$

Note that the solution is independent of θ in these circumstances. A simplified solution, independent from a and θ, for $c_r^*(a,\theta)$ was also obtained. Using (5.11) the result is

$$c_r^*(a,\theta) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2} \qquad (6.5).$$

Combining these results, (6.4) and (6.5), as in (5.9) it is observable that the asymptotic approximation for this particularization reduces to $w_r^*(t;a,\theta) \approx v_r^*(a,\theta) - \pi_r(t)$, where the function $\pi_r(t)$ is, considering (6.4) and (6.5),

$$\pi_r(t) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}\, e^{-\frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2+2r\sigma^2}\right)^2}{2\sigma^2}t}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \qquad (6.6).$$

7 The Assets and Liability Behavior Representation

In this section it is presented an application of the results obtained above to an asset-liability management scheme of a pension's fund. Assume that the assets value process of a pensions fund may be represented by the geometric Brownian motion process

$A(t) = be^{a+(\rho+\mu)t+\sigma B(t)}$ with $\mu < 0$ and $ab\rho + \mu\sigma > 0$, where $B(t)$ is A standard Brownian motion process. Suppose also that the fund liabilities value process performs such as the deterministic process $L(t) = be^{\rho t}$.

Consider now the stochastic process $Y(t)$ obtained by the elementary transformation of $A(t)$, $Y(t) = \ln\frac{A(t)}{L(t)} = a + \mu t + \sigma B(t)$.

This is a generalized Brownian motion process exactly as the one studied before, starting at a, with drift μ and diffusion coefficient σ^2. Note also that the firs passage time of the assets process $A(t)$ by the mobile barrier T_n, the liabilities process, is the first passage time of $Y(t)$ by 0-with finite expected time under the condition, stated before, $\mu < 0$.

Consider also the pensions fund management scheme that raises the assets value by some positive constant θ_n, when the assets value falls equal to the liabilities process by the n^{th} time. This corresponds to consider the modification $\overline{A}(t)$ of the process $A(t)$ that restarts at times T_n when $A(t)$ hits the barrier $L(t)$ by the n^{th} time at the level $L(T_n) + \theta_n$. For purposes of later computations, it is a convenient choice the management policy where

$$\theta_n = L(T_n)\left(e^{\theta} - 1\right), \text{for some } \theta > 0 \quad (7.1).$$

The corresponding modification $\tilde{Y}(t)$ of $Y(t)$ will behave as a generalized Brownian motion process that restarts at the level $\ln \frac{L(T_n)+\theta_n}{L(T_n)} = \theta$ when it hits 0 (at times T_n).

Proceeding this way, it is reproduced via $\tilde{Y}(t)$ the situation observed before when the Brownian motion example was treated. The Laplace transform in (6.1) is still valid.

Similarly, to former proceedings, the results for the present case will be distinguished with the symbol $(\#)$. It is considered the pensions fund perpetual maintenance cost present value, because of the proposed asset-liability management scheme, given by the random variable: $V^{\#}(r,a,\theta) = \sum_{n=1}^{\infty} \theta_n e^{-rT_n} = \sum_{n=1}^{\infty} b\left(e^{\theta} - 1\right)e^{-(r-\rho)T_n}, r > \rho$, where r represents the appropriate discount interest rate. To obtain the above expression it was only made use of the $L(t)$ definition and (7.1). Note that it is possible to express the expected value of the above random variable with the help of (6.2) as

$$v_r^{\#}(a,\theta) = \frac{b\left(e^{\theta} - 1\right)}{\theta} v_{r-\rho}^{*}(a,\theta) = \frac{b\left(e^{\theta} - 1\right)e^{-K_{r-\rho}a}}{1 - e^{-K_{r-\rho}\theta}} \qquad (7.2).$$

As $\theta \to 0$

$$\lim_{\theta \to 0} v_r^{\#}(a,\theta) = \frac{be^{-K_{r-\rho}a}}{K_{r-\rho}} \quad (7.3).$$

Another expression that may be found in Gerber and Parfumi (1998).

In a similar way, the maintenance cost up to time t in the above-mentioned management scheme, is the stochastic process $W^{\#}(t;r,a,\theta) = \sum_{n=1}^{N(t)} b\left(e^{\theta} - 1\right)e^{-(r-\rho)T_n}$, $W^{\#}(t;r,a,\theta) = 0$ if $N(t) = 0$, with expected value function

$$w_r^{\#}(t;a,\theta) = \frac{b\left(e^{\theta} - 1\right)}{\theta} w_{r-\rho}^{*}(t;a,\theta) \qquad (7.4).$$

The results of section 6 with r replaced by $r - \rho$ may be combined as in (7.4) to obtain an asymptotic approximation.

8 Conclusions

This chapter presents a stochastic processes tool to study the maintenance costs of a pension's fund, supporting the fund managers and contributing for an adequate decisions planning. In general diffusion setting, the main results are formulae (4.1) and (5.9). The whole work depends on equation (3.1) solvability, in order to obtain the first passage times Laplace transforms. But the known solutions happen only in very rare cases. An obvious case, for which the equation solution is available, is the Brownian motion diffusion process. The main results concerning this particularization are formulae (6.2) and (6.6). Certain Brownian motion process transformations, that allowed to make use of the available Laplace transform, may be explored as it was done in section 7. Formulae (7.2) and (7.4) are this case most relevant results.

References

Andrade M, Ferreira MAM, Filipe JA (2012) Representation of reserves through a Brownian motion model. SPMCS 2012. JPCS-Journal of Physics: Conference Series 394 (2012) 012036. IOP Publishing

Andrade M, Ferreira MAM, Filipe JA, Coelho MP (2012) The study of a pensions fund equilibrium through an infinite servers nodes network. International Journal of Academic Research Part A, 4(3):205-208

Bass RF (1998) Diffusions and Elliptic Operators. Springer-Verlag, New York, U. S. A.

Bhattacharya RN, Waymire E (1990) Stochastic Processes with Applications. John Wiley & Sons, New York, U. S. A.

Feller W (1971) An Introduction to Probability Theory and its Applications (vol. II, 2^{nd}ed.). John Wiley & Sons, New York, U.S.A.

Ferreira MAM (2012) Non-autonomous pensions funds maintenance costs study through a diffusion process. International Journal of Academic Research Part A 4(6):51-5. doi:10.7813/2075-4124.2012/4-6/A.7

Ferreira MAM, Andrade M, Filipe JA, Coelho MF (2011) Statistical queuing theory with some applications. International Journal of Latest Trends in Finance and Economic Sciences 1(4):190-195

Ferreira MAM, Andrade M, Filipe JA (2012) Studying pensions funds through an infinite servers node network: a theoretical problem. SPMCS 2012. JPCS-Journal of Physics: Conference Series 394 (2012) 012035. IOP Publishing

Figueira J (2003) Aplicação dos processos de difusão e da teoria do renovamento num estudo de reservas aleatórias. Ph.D. Thesis presented at ISCTE, Lisboa, Portugal

Figueira J, Ferreira MAM (2003) Cost of non-autonomous pensions funds via an application of diffusion processes. Review of Financial Markets 5(1):39-50

Filipe JA, Ferreira MAM, Andrade M (2012) Reserves represented by random walks. SPMCS 2012. JPCS-Journal of Physics: Conference Series 394 (2012) 012034. IOP Publishing

Gerber HU, Parfumi G (1998) Stop-loss a tempo continuo e protezione dinamica di un fondo d'investimento. Revista di Matematica per le Scienze Economiche e Sociale 21:125-146

Impavido (2012). Pension Funds. In: Gerard Caprio Jr, Douglas W. Arner, Thorsten Beck, Charles W. Calomiris, Larry Neal, Nicolas Veron (Eds.), Handbook of Key Global Financial Markets, Institutions and Infrastructure. Elsevier

Karlin S, Taylor H (1981) A Second Course on Stochastic Processes. Academic Press, New York, U. S. A.

Refait C (2000) Default risk estimation and stochastic calculus: Application to French industrial firms. 4^{th} International Congresso in Insurance and Mathematics, Barcelona, España.