Losers distribution, with applications to financial inclusion

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Losers distribution, with applications to financial inclusion: *Lightning can strike twice, but it may not strike at all*

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Abstract

In this paper I develop the new "Losers Distribution" and use it to build a financial instrument which partially solves the problem of financial exclusion. I design compensated lotteries that can mobilize and circulate the savings of people who do not participate in formal financial markets. The lottery which I propose entirely avoids problems of information asymmetry, default risk, and ethnic and gender discrimination in lending. I use the losers distribution to set lottery ticket prices so that only the financially excluded participate and to calculate the subsidy needed to sustain the lottery. I show that the lottery subsidy rate is low when compared to the subsidies of typical microfinance programs about the world.

Keywords: Financial Exclusion, Inequality, Financial Market Failure, Income Distribution, Poverty, Lotteries

JEL codes: O1, C1, G1, H4,

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1 Introduction

1.1 Summary

In this paper I make two significant contributions. I develop a new discrete probability distribution —which I call the "losers distribution"— that has applications in number theory and in multinomial choice modeling. I also design the "compensated lottery" as a new financial instrument which partially solves the problem of financial exclusion. I link these two contributions by using the losers distribution to design the compensated lottery pricing. I show that lotteries with a high probability of winning small prizes can be designed to play the role of financial instruments. They can be priced to only appeal to people who do not participate in formal financial markets. This means that the lottery mobilizes new financial resources —the savings of the poor— and that it does not crowd out formal lending —from the savings of the wealthy. The lottery I propose offers compensation in the form of a guaranteed prize after a fixed number of trials. The losers distribution is needed to calculate the *ex post* income distribution, the number of losers who will have to be compensated, and the optimal subsidy.

In the next section I propose and develop the losers probability distribution and explore some of its properties. I also show how it can be calculated directly and recursively and show the relation between the losers distribution and well know distributions and number series such as the Bose-Einstein statistic, Pascal's triangle, Stirling, Bell, and Fubini numbers. In Section 3 I discuss the problem of financial exclusion and in Section 4 I study the use of lotteries as financial instruments for the poor. I propose the compensated lottery in section 5 and derive the ticket price and subsidy rates that make it work. Section 6 provides the conclusions.

2 MODEL

Many events arise that have a finite number of equally probable and mutually exclusive possible outcomes. Yet even after many trials not every possible outcome is expressed. Some may not occur even if the number of trials significantly exceeds the number of possible outcomes. One example is the repeated roll of a fair die. The frequency with which each side shows up after a given number of rolls is tallied. After 10 rolls no six or no four, or no six *and* no four turn up. A second example is a game of chance where a group of people each put some money into a pot. A game takes place and one winner takes the pot. The game is repeated a pre-determined number of times. Some players may never win, even if the number of games they play is fairly large. Example three involves a new, multi-unit housing development with identical homes that has just been put on the market. Prospective buyers are allowed to inspect a home of their choosing. After one week, some homes have been seen more than once and some may have not been visited at all.

These relations are all non-injective; more than one element of the domain of die rolls, games, and home visits can map onto the same outcome element in the codomains. But the more important property for the purposes of this paper is that the relations are also non-surjective –not every element in the codomain of outcomes (die-sides, game players, or houses) will necessarily be manifested. For ease of exposition, the set of outcomes that are not manifested is called "the set of losers", or just "losers." The question addressed here is *what is the distribution that characterizes the number of losers when trials are identical and all outcomes are equally likely*?

2.1 Lottery Games

In this section I will introduce the "losers distribution," a new discrete probability distribution that describes the probability that after n identical trials there will be l losers and w winners. We have $l + w = k$ —where k is the number of players –and

$$\sum_{i=1}^{k} w_i = n.$$

The expected number of losers is calculated. The paper ends with an application to financial exclusion and a proposal to extend financing to the poor.

A *"lottery game"* consists of a set N of identical random draws[1] with replacement that has cardinality $n = |N|$ and elements indexed by $i \in \{1, 2, ..., n\}$. The elements of N are labeled. To simplify the exposition and the transition to the application at this end of the paper, I will refer to the domain set of the game as *"lotteries"* and the co-domain as *"players"*. A non-injective, non-surjective relation maps the set of lotteries to the set of players. Without loss of generality, we can think of lottery draws as occurring sequentially with the index representing the order in which the lotteries take place. The co-domain consists of a set K of lottery players with cardinality $k = |K|$, whose players are indexed by $j \in \{1, 2, ..., k\}$. The players are labeled. The sample space of a game is thus k^n. For most of the discussion here it will be assumed that $n \geq k$.

Every lottery in an n-game has k outcomes; it assigns a win to a player j and losses to each of the remaining $(k-1)$ players. These lottery outcomes can be represented by a $kx1$ column vector with $(k-1)$ zeros for the losers and the number one in position j for the winner. The set of possible win vectors for a 5-person game are shown in Figure 1. Columns 1 through 5 represent wins by players 1 through 5, respectively. For instance, the second 5−element column vector –with a 1 as its second element and zeros

[1] The analysis can be extended to allow for some outcomes to be more likely than others. This context can be handled within the framework developed here, but for the remainder of this paper it is assumed that all possible outcomes are equally likely, with probability $p = \frac{1}{n}$. Extending the analysis to contexts with unequal values of p has interesting applications to multinomial choice modelling —currently a cumbersome set of tools. However this extension is left for later work.

Figure 1: **Possible Win Vectors for a 5−Person Game**

$$\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

elsewhere –represents the outcome of a 5−person lottery draw where player 2 was the winner and player 1 and players 3 through 5 lost. If the probability of being drawn is *not* equal for all players, copies of the more likely columns can be added to this set to reflect the population from which unequal relative frequencies are drawn. However, for the remainder of this paper I will assume that the probability of winning is $\frac{1}{n}$, the same for all of the players.

Figure 2: **A 5-Player, 5-Game Win Matrix**
Player 1 and 2 win twice and player 3 wins the last game

$$\begin{bmatrix}1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The collection of win-vectors from a realized lottery game can be arranged into a "win-matrix"' to represent the full set of outcomes from the game. The column sequence represents the order in which wins took place. *Figure* 2, illustrates one possible outcome of a 5-player, 5-game lottery in which player 1 won the first and third draws, player 2 won the second and fourth draws, player 3 won the last draw, but players 4 and 5 each had zero wins.

Each lottery in a game randomly selects one of the columns in Figure 1. The selected column is copied and then appended from left to right onto a new *kxn win-matrix "**W**"*. Once the game concludes, the order of columns in $\mathbf{W}$ records the sequence of outcomes while the sum of each of the k rows gives the final tally of wins for players 1 through k. Thus post-multiplying $\boldsymbol{W}$ by an $nx1$ unit vector $\boldsymbol{U}$, results in a $k \times 1$ column vector $\mathbf{R}'$ which tallies the wins. The transpose of this vector $R = (w_1, w_2, w_3, \ldots w_k)$ is a convenient way to represent the distribution of winnings from the game.

The *Figure* 3 illustrates one such tally for a 5−player, 9−lottery game in which player 1 won the first four times, player two won lotteries 5 and 6, player 3 won the last 3 times. Players 4 and 5 lost every time.

The distribution of winnings from the game can be more conveniently represented by the transpose of the right hand column vector $\mathbf{R} = (4, 2, 3, 0, 0,)$. Commas have been inserted for clarity. Notice that while $\mathbf{R}$ conveniently tallies the wins, it loses all the information about the sequence of lotteries that produced

Figure 3: **Tallying the wins of a 5-player, 5-fold lottery game** ($k = 5$, $n = 9$)
$\mathbf{W}_{5\times 9} \times \mathbf{U}_{9\times 1} = \mathbf{R}'_{\mathbf{5\times 1}}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{w}_1 \\ \mathrm{w}_2 \\ \mathrm{w}_3 \\ \mathrm{w}_4 \\ \mathrm{w}_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

them. In fact the columns in $\mathbf{W}$ can be arranged in up to $n!$ different ways to produce the same $\mathbf{R}$ vector.

We will return to this fact shortly, with more precision. For now, notice that (4,2,3,0,0) is a partition of the number 9. In fact, since n and the w_i are integers, with $w_i \leq n$ and

$$\sum_{i=1}^{k} w_i = n,$$

every $\mathbf{R}$-vector describes a partition of the number n into k parts (counting zeros as parts). Partition theory typically ignores zeros –in what are called *proper* partitions –and treats the parts into which a number is divided as anonymous, indistinguishable, and interchangeable summands. Thus the $\mathbf{R}$-vectors $(4, 2, 3, 0, 0, 0, 0, 0, 0), (4, 2, 3, 0, 0), (0, 2, 3, 4)$, and $(4, 3, 2)$ result in the same traditional partition of 9 into 3 parts.

So called *proper partitions* ignore zeros, and are conventionally ordered from their largest to smallest elements. In other words partition theory is concerned with sums of the form $\sum_{i=1}^{w} w_i = n$, where $w_i \geq 1$, $\forall i$, and $i{>}j \implies w_i \geq w_j$. For instance $(7, 1, 1), (6, 2, 1), (5, 3, 1), (5, 2, 2), (4, 4, 1), (4, 3, 2)$, and $(3, 3, 3)$ are the seven proper partitions of the number nine into 3 parts. In standard notation the number of partitions of n into w parts is written as $p(n, w)$, and the total number of partitions of the number n as $p(n)$, where $p(n){=}\sum_{i=1}^{w} p(n, w)$. So, continuing the earlier example, there are 7 distinct partitions of the number 9 into 3 parts —so $p(9, 3) = 7$— and there are altogether 30 partitions of the number 9: $p(9) = 30$. The number of partitions grows very rapidly with n: $p(3) = 3, p(9) = 30, p(20) = 627, p(50) = 204, 226$, and $p(100) = 190, 569, 292$ (sequence $A000041$).

The theory of partitions is a well-developed subfield of number theory (for instance Andrews, 1984). But we must modify standard partition notation for our problem, because the arrangements of a partition matter in a lottery game, as do the zeros! In lottery game score-keeping, every ordering of a given proper partition represents a different allocation of the total n wins among w winners. For instance, the partition $(4, 3, 2)$

of the number 9 has $3! = 6$ versions that represent different allocations of a $4-3-2$ split of wins among three winners of a nine-lottery game. The number of allocations that are possible grows when losers are also accounted for.

Consider the case where $n = k$ and every player wins once, that is $w_i = 1\ \forall i \in \{1, 2, \ldots, k\}$. There are $n!$ different draw sequences that can result in each player winning once, and yet all of these outcomes would be represented by the same **R**-vector of ones. This is because **R**-vector notation loses information on the sequence in which wins take place. This is a large number to ignore. Economists will also readily recognize that the sequence itself matters. If the intervals between lotteries in a game are significant it is not the same to be the first winner as to be the last one even if the nominal value of the prize is the same. In a 9-player, 9-lottery game there are $9! = 362,880$ different sequences that can lead to each player winning once. If all players win at least once and $n > k$, then some of the columns in the win-matrix $\mathbf{W}$ must be repeated for players who won more than once. An example of this can be visualized by ignoring the bottom two rows of zeros of the $\mathbf{W}$ matrix in *Figure* 3, which then represents a 3-person, 9-fold lottery where nobody lost. To find the number of unique permutations of $\mathbf{W}$, the $n!$ arrangements of columns must be divided by the product of the $w_i!$ in order to account for identical columns. Thus the number of permutations of $\mathbf{W}$ is

$$\frac{n!}{w_1!, w_2!, w_3!, ..., w_k!}, \tag{1}$$

which is of course a multinomial coefficient. Multinomial coefficients ignore the $l = n - w$ losers, just as partition notation ignores zero summands. Since $0! = 1$, an arbitrary number of players with zero wins can be added without changing the numerical value of the multinomial coefficient. So what the coefficient actually counts is the number different ways that a given distribution of wins —a given **R**-vector— can be drawn *among the subset of players who win at least once*: the winners. This multinomial coefficient is not the correct basis for calculating the probability that this turns out to be the number of winners.

2.2 THE LOSERS DISTRIBUTION

The discussion to this point has laid out the reasoning and rationale behind each step in the construction of $L(l \mid n, k)$ –the frequency with which l losers will arise in a k-player, n-fold lottery game. Since the sample space is known to be to be k^n, it is straightforward to calculate the probability of these events once $L(l \mid n, k)$ is know as $Pr\,(l \mid n, k) = \frac{L(l|n,k)}{k^n}$. In the example of a 9-player, 9-fold lottery game shown in Appendix Table A1, we obtain $Pr(l = 3 \mid n = 9, k = 9) = \frac{L(3|9,9)}{9^9} \approx 0.41$. The probability that 3 out of 9 players won't win anything at all is about 41% .

<u>**Definitions**</u>: : Let K be a set of players and N be a set of lotteries, with cardinalities $k = \|K\|$ and $n =$

$\|N\|$, respectively. Index the elements of K as $i \in \{1, 2, \ldots, k\}$ and the elements of N as $j \in \{1, 2, \ldots, n\}$. A lottery game is a non-surjective, non-injective relation with domain N and codomain K that assigns each lottery to one player. The assignments are called "wins". After the game, player i has w_i wins, with $w_i \geq 0$ $\forall$ i and $\sum_{i=1}^{w} w_i = n$. Define the player subset $\mathfrak{W}$ consisting of all players for whom $w_i \geq 1$. Players in $\mathfrak{W} \subseteq K$ are called "winners" and $\mathfrak{W}$ has cardinality $w = \|\mathfrak{W}\|$. Players with zero wins are "losers" and form the complement set $\mathfrak{L}$ with cardinality $\boldsymbol{l} = \|\mathfrak{L}\|$, so $\mathfrak{L} \cup \mathfrak{W} = K$, $\boldsymbol{l} + \boldsymbol{w} = n$, and $0 \leq \boldsymbol{l} \leq (k-1)$.

The tally of wins is arranged into a win vector $\mathbf{R} = (w_1, w_2, \ldots, w_w)$. When the losers are ignored and the $\mathbf{R}$-vector elements are arranged from largest to smallest, they form a proper w-fold partition of n. Let $\sigma_{n,w}$ designate one such w-fold partition of n and $\mathfrak{S}_{n,w}$ the set of all w-fold partitions of n.

As we have seen, traditional partition notation arranges partition summands from largest to smallest, so that in $\mathbf{R} = (w_1, w_2, \ldots, w_w)$, $i > j$ implies $w_i \geq \mathrm{w}_j$. For instance, one of the partitions of 9 into 6 parts would be written as $(2, 2, 2, 1, 1, 1)$. When $n!$ is divided by the factorials of all elements in a given $\sigma_{n,w}$, as in $M_1 = \frac{n!}{\prod_{i=1}^{w} w_i!}$, the resulting quotient is the multinomial coefficient which counts the ways in which a particular $\sigma_{n,w}$ can arise as the win-vector of an n-fold lottery with w players.

There is an alternative partition notation: $\mathbf{R} = (w_1^{r_1}, w_2^{r_2}, \ldots w_m^{r_m})$. In this case the partition has m distinct parts. *Distinct* w_i are ordered from largest to smallest and the r_i coefficients indicate how many times summand w_i is repeated in the partition; these coefficients should not be confused with exponents. Thus $\mathbf{R} = (2, 2, 2, 1, 1, 1)$ can be written more succinctly in this alternative notation as $\mathbf{R} = (2^3, 1^3)$. This notation allows us to define a second multinomial coefficient as $M_2 = \frac{w!}{\prod_{i=1}^{m} r_i!}$. M_2 counts how many different ways a given $\mathbf{R}$-vector split can be arranged —thus treating the partition $(2, 1, 2, 2, 1, 1)$ as different from $(2, 2, 2, 1, 1, 1)$, because it has the same parts, but they are arranged in a different order.

The frequency with which a particular split arises —what in thermodynamics is called the "multiplicity" of a state —is $M_1 \times \mathrm{M}_2$ —the product of these two multinomial coefficients.

$$Freq\left[\mathbf{R}\left(w_1, w_2, ..., w_w\right)\right] = M_1 \times M_2 = \frac{n!}{\prod_{i=1}^{w} w_i!} \frac{w!}{\prod_{i=1}^{m} r_i!}. \tag{2}$$

A result related to (2) is discussed in Faris (2011). *Equation* (2) gives all of the ways that n labelled objects can be assigned to w labeled sets in such a manner that none are empty. Kao and Zetterberg (1957) Theorem 2.2 proves that the number of multinomial coefficients that can be formed from the proper partitions of n into w parts —designated as $Q(n, w)$ in this paper— is given by

$$Q(n, w) = \sum_{\sigma_{n,w} \in \mathfrak{S}} \frac{n!}{\prod_{i=1}^{w} w_i!} = \sum_{i=0}^{w-1} (-1)^i \binom{w}{i} (w-i)^n. \tag{3}$$

$Q(n,w)$ counts all of the ways in which n labeled objects can be assigned to w labeled sets, such that no set is empty. This contrasts with Stirling numbers of the second kind, $S(n,w)$, which count all the ways in which n labeled objects can be assigned to w unlabeled sets:

$$S(n,w) = \frac{1}{w!}\sum_{i=0}^{w-1}(-1)^i\binom{w}{i}(w-i)^n = \frac{1}{w!}Q(n,w)\,. \tag{4}$$

Equations (3) and (4) show that $Q(n,w)$ numbers are related to Stirling numbers by a factor of $w!$. Appendix Table A2 shows the first 10 rows of $S(n,w)$ and their relation to the Bell numbers. Tables A2 through A4 give examples of the mechanical derivation of the Q numbers for $Q(1,1)$ through $Q(10,10)$.

We are now in a position to characterize the probability mass function of the losers distribution.

Theorem 1 Let $k, n \in \mathbb{Z}^+$ and $n \leq k$. The probability of $\boldsymbol{l}$ losers in a k-player, n-fold lottery game is given by

$$Pr(\boldsymbol{l} \mid n,k) = \frac{\binom{k}{l}\sum_{i=0}^{k-l-1}(-1)^i\binom{k-\boldsymbol{l}}{i}(k-\boldsymbol{l}-i)^n}{k^n} \tag{5}$$

Proof. Clearly $0 \leq \boldsymbol{l} \leq (k-1)$, so if there are $\boldsymbol{l}$ losers there are $w = k - \boldsymbol{l}$ winners. Winners distribute the n wins among themselves in w parts. These parts form a proper partition of n. Letting $\sigma_{n,w}$ represent one such w-fold proper partition of n and $\mathfrak{S}_{n,w}$ represent the set of all proper w-fold partitions of n, from Kao and Zetterberg (1957) Theorem 2.2, we have that

$$\sum_{\sigma_{n,w}\in\mathfrak{S}_{n,w}} \frac{n!}{\prod_{i=1}^{w} w_i!} = \sum_{i=0}^{w-1}(-1)^i\binom{w}{i}(w-i)^n\,. \tag{6}$$

Equation (6) gives the frequency with which a w-fold split of the lotteries will occur, with w players and every player winning at least once —it ignores the possibility that there might be one or more losers. But given any w-fold split of wins from a k-player, n-lottery game, $\boldsymbol{l}$ zeros can be added to the win-vector $\mathbf{R}$ in $\binom{w+l}{\boldsymbol{l}}$ ways. The frequency with which a w-fold split will occur in a k-person, n-lottery game —and thus $L(\boldsymbol{l})$, the frequency with which there will be $\boldsymbol{l}$ losers is therefore

$$L(\boldsymbol{l}) = \binom{k}{\boldsymbol{l}}\sum_{\sigma_{n,w}\in\mathfrak{S}} \frac{n!}{\prod_{i=1}^{w} w_i!} = \binom{k}{\boldsymbol{l}}\sum_{i=0}^{w-1}(-1)^i\binom{w}{i}(w-i)^n. \tag{7}$$

Letting $w = k$-$\boldsymbol{l}$ and dividing the frequency by the sample space gives the probability of $\boldsymbol{l}$ losers $Pr(\boldsymbol{l} \mid n,k)$ $= \frac{\binom{k}{l}\sum_{i=0}^{k-l-1}(-1)^i\binom{k-l}{i}(k-\boldsymbol{l}-i)^n}{k^n}$, as in Equation 5. ■

Corollary 1: The expected number of losers in a k-player, n-fold, lottery game is given by

$$\hat{l} = E(\boldsymbol{l}) = \frac{\sum_{l=0}^{k-1} \boldsymbol{l} L(\boldsymbol{l})}{k^n}. \tag{8}$$

Corollary 1 is straightforward. Since L($\boldsymbol{l}$) is the frequency with which $\boldsymbol{l}$ losers will arise, dividing this by the sample space k^n generates the *probability* that $\boldsymbol{l}$ losers will arise. The sum in (8) is the probability weighted, convex combination of all possible values of $\boldsymbol{l}$.

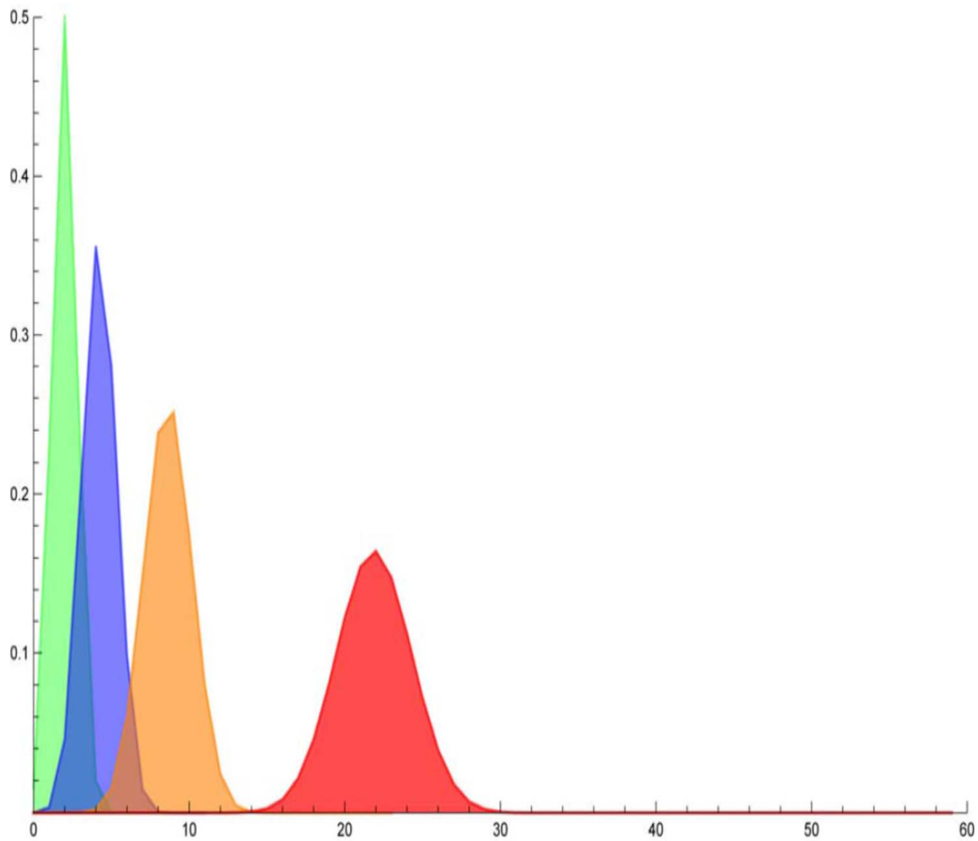

Figure 4: Probability Mass Functions for the Losers Distribution
(For $n \in \{6, 12, 24, 60\}$)

Figure 4 plots losers distribution probability mass functions for lottery games of size $n = 6, 12, 24$, and 60. The pmf's are uni-modal and right-skewed. Since the mean converges to about one-third of n and the upper bound on the number of losers is $(n-1)$, the probability mass function has a long right tail that grows as n grows.

Table 1 shows the mean, variance, standard deviation, and coefficient of variation for losers distributions

Table 1: **Losers Distribution: Descriptive Statistics for Selected Values of n**

n	Mean	Variance	St. Dev.	Coeff. of Var.	Skewness
1	0.00	0.000	0.00	$--$	$--$
6	2.01	0.605	0.79	0.387	0.778
12	4.22	0.993	1.00	0.236	1.229
24	8.64	2.353	1.53	0.177	1.773
60	21.89	5.852	2.42	0.111	2.551
120	43.96	11.684	3.42	0.078	3.217
500	183.76	48.624	6.97	0.038	4.907
1000	367.70	97.228	9.860	0.027	5.936

with $n = k$ and $n \in \{1, 6, 12, 24, 60, 120, 500, 1000\}$. The mean rises monotonically with n but the ratio $\frac{\hat{1}}{n}$ converges asymptotically to the multiplicative inverse of Euler's number —i.e., to approximately 0.368 (see Figure 5). Similarly, the variance and standard deviation of the losers distribution also rise, but at a much slower rate than n. This is shown by the coefficient of variation in Table 1, which falls from 0.387 for $n = 6$ to 0.027 for $n = 1,000$.

The losers distribution is right-skewed and plots in Figure 4 suggest that this asymmetry grows with the size of a lottery game n. The coefficient of skewness for selected values of n —shown in the right-most column of Table 1— confirm this impression. Skewness rises monotonically with n, though at a decreasing rate. When game size doubles from $n = 12$ to $n = 24$, the coefficient of skewness rises by more than 44%, but when game size doubles from 500 to 1,000, skewness rises by less than 21%.

2.3 Recursive Relations

There is a recursive relation between the $Q(n, w)$.

<u>Theorem 2</u>: Let $Q(n, w)$ indicate the sum of all multinomial coefficients generated by the proper partition of n into w parts. There exists a recursive relation of the form

$$Q(n, w) = w\{Q[(n-1), (w-1)] + Q[(n-1), w]\}. \tag{9}$$

Proof. From equation (3) we have that the term in brackets in (7) becomes

$$Q[(n-1), (w-1)] + Q[(n-1), w] =$$
$$\sum_{i=0}^{w-1} (-1)^i \binom{w}{i} (w-i)^{n-1} + \sum_{i=0}^{w-2} (-1)^i \binom{w-1}{i} (w-1-i)^{n-1}$$

The first term can be rewritten to obtain

$$\binom{w}{0}w^{n-1} + \sum_{i=1}^{w-1}(-1)^i\binom{w}{i}(w-i)^{n-1} + \sum_{i=0}^{w-2}(-1)^i\binom{w-1}{i}(w-1-i)^{n-1} =$$
$$w^{n-1} + \sum_{i=1}^{w-1}(-1)^i\binom{w}{i}(w-i)^{n-1} - \sum_{i=1}^{w-1}(-1)^i\binom{w-1}{i-1}(w-i)^{n-1}.$$

The sign change for the third term in the last equation comes from changing its limits of summation. Collecting terms and making use of the Pascal triangle identity $\binom{w}{i} = \binom{w-1}{i} + \binom{w-1}{i-1}$, this becomes

$$w^{n-1} + \sum_{i=1}^{w-1}(-1)^i\binom{w-1}{i}(w-i)^{n-1}$$

$$= \sum_{i=0}^{w-1}(-1)^i\binom{w-1}{i}(w-i)^{n-1}$$

$$= \sum_{i=0}^{w-1}(-1)^i\binom{w-1}{i}\frac{(w-i)^n}{(w-i)}$$

When this last expression is multiplied by w, we obtain

$$\{Q[(n-1),(w-1)] + Q[(n-1),w]\}w = \{\sum_{i=0}^{w-1}(-1)^i\binom{w-1}{i}\frac{(w-i)^n}{(w-i)}\}w =$$
$$\sum_{i=0}^{w-1}(-1)^i\binom{w}{i}(w-i)^n = Q(n,w),$$

the desired result. ■

Comments. The recursive nature of $Q(n,w)$ lends itself to the construction of a Q-triangle, shown in $Table\ A4$, that is analogous to Pascal's triangle. Once the values for the n^{th} row are known, nothing else is needed in order to construct row $(n+1)$. This can be verified in Appendix Table A4 which provides Q-triangle values $n \leq 10$. Interestingly, the row sums of the Q-triangle form the Fubini numbers —also known as the *ordered* Bell numbers (Series $A000670$).

The Q-triangle can also be derived as a series of Hadamard, or entrywise matrix products. The required sequence of such lower triangular matrices (with the upper triangles of zeros omitted) appears in Tables $A2$ through $A6$. Entries of the matrix in $Table\ A2$ are Stirling numbers of the second kind, $S(n,w)$ —series *A008277*. They count the ways in which n distinguishable, or "labeled" elements of objects can be assigned to w indistinguishable, or "unlabeled" sets. Stirling matrix row sums $\sum_{w=1}^{n} S(n,w)$ yield the Bell numbers —series *A000110*.

$Table\ A3$ contains the permutations $w_i!$ for $w_i \leq n$. As *Equation* (3) makes clear, the Q-Triangle entries $Q(n,w)$ of $Table\ A4$ can be derived as $w!S(n,w)$. So, letting $\mathfrak{M}_i$ stand for the lower-triangular matrix in $Table$ "i", the elements of $Table$ 4 are obtained as $\mathfrak{M}_2 \diamond \mathfrak{M}_3 = \mathfrak{M}_4$, where $\diamond$ stands for the Hadamard product. The elements in row n, column w that lie on or below the diagonal of $\mathfrak{M}_4$ form the Q-triangle, the sum of all multinomial coefficients that can be generated from a proper partition of n into w parts. The row sums of this matrix are given by

$$T\sum_{w=1}^{n} Q(n,w) = \sum_{w=1}^{n} w!S(n,w) \tag{10}$$

and generate the Fubini numbers —series *A000670*, also known as the *ordered* Bell numbers. $Table\ A5$ lists the left-justified, binomial coefficient values of Pascal's triangle —omitting the first column of $\binom{n}{0} = 1$ values. These have rows which sum to Gaussian binomial coefficients. Finally, $Table\ A6$, shows the "Losers distribution" matrix. It consists of a lower-diagonal matrix whose elements are the Hadamard product of corresponding Q-matrix elements of $Table\ A4$ and binomial coefficients from the Pascal triangle (shown in $Table\ A5$): $\mathfrak{M}_6 = \mathfrak{M}_4 \diamond \mathfrak{M}_5$, or equivalently $\mathfrak{M}_6 = \mathfrak{M}_2 \diamond \mathfrak{M}_3 \diamond \mathfrak{M}_5$. Note that the rows in every row n of $Table$ $A6$ sums to n^n, the sample space, proving that this is indeed a probability distribution function.

In the comming sections I develop an application of the losers distribution to an important problem in development economics, the study of poverty, and the determinants of income distribution.

3 APPLICATION TO FINANCIAL INCLUSION

It is well known that the poor are less likely to participate in formal financial markets (Morduch, 1999). In times of economic shock or when faced with short-term cash flow problems, the poor typically turn to informal sources of finance, including friends and family, pawn shops, moneylenders, and loan sharks. This empirical regularity has spurred the search for mechanisms to bridge this "financial exclusion". Muhammad Yunus' work in Bangladesh is one of the earliest and the best known efforts to formally address this unmet demand of the poor for financial instruments by building social capital among small groups of borrowers. Grameen Bank's "micro-finance" scheme met with great early success and acclaim, earned Yunus a Nobel Peace Prize, and spun off many emulators. The last two decades have seen a surge of creativity and financial resources invested in developing more structured and less usurious ways for "financial inclusion" in formal credit markets such as organizing credit cooperatives and rotating savings and credit associations (Armendáriz de Aghion and Murdoch, 2005).

Sadly, recent assessments have been less sanguine about the mechanisms behind success of Grameen Bank lending (Jain, 1996), about the long term benefits of micro-lending in general (Banerjee, Duflo, Glennerster, and Kinnan, 2015 and Banerjee, Karlan, and Zinman, 2015), raised the possibility of market power (Bairagi et al., 2014), and even cautioned that microfinance coverage expansion may be a primrose path leading to large scale collapse of the formal banking sector (Polgreen and Vajaj, 2010). The problem is that informal lenders who want to lend to the financially exluded poor must ultimately confront the same hurdles that are faced by formal sector lenders. Informal lenders face the same problems of information asymmetry, absence of collateral, and lack of experience and reputation which jointly translate into an inability to asses and charge for differential default risk.

Financial exclusion is not confined to developing countries. It is a concern in well-developed financial markets too. The creation of the Consumer Financial Protection Bureau (CFPB) through the Dodd–Frank Wall Street Reform and Consumer Protection Act of 2010 was a formal —if belated— recognition of the exclusion problem in U.S. financial markets. In 2016, the CFPB conducted a "Financial Wellbeing Survey", one of the first of its kind. This 217-question, nationally representative survey of 6,394 adults confirms that a large segment of the U.S. population is unserved by formal financial markets. About 22% of respondents saw themselves as financially excluded in that they had applied for credit in the previous year and were turned down, or did not apply because they thought that they would be turned down. Yet, when asked if in times of hardship they have friends or family who would lend them money and expect repayment, 39% of American adults replied "no". When asked if family or friends would lend them money without expecting repayment, 68% said "no". About 4.5% used payday loans, cash advance loans, or pawn shops to obtain loans. Respondents were asked if they would prefer to receive \$816 today or \$860 in 3 months. Among the non-excluded 64.3% preferred to wait for \$860 in 3 months, but 60.2% of the financially excluded preferred \$816 today —implying an annual discount rate of 23.3%![2].

A large fraction of the American population is excluded from the formal financial market. In analyzing the results of a Federal Reserve Board survey of small businesses, Cole and Wolken (1995) find that four in ten small business owners in the U.S. used their personal credit cards to finance company debt. This practice was significantly more common among new companies and among companies with African-American owners. They report that "*credit card balances usually carry much higher interest rates than do commercial loans of comparable size*" (637) and require full payment of outstanding balances each month. Campbell et al. (2011) report that payday loans charge interest in the range of 15 to 30% *per week* and that the loans are small —with 80% below \$300. Lawrence and Elliehausen (2008) find that payday loan borrowers tend to be less educated, younger, less likely to own a home, and have poor credit ratings —73% have been turned down for formal credit within the past 5 years, compared to the national average of 21.8% of all adults.

[2]These statistics are the author's calculations, using the CFPB survey data.

3.1 Financial Self-Exclusion

The reason for exclusion explored in this section is voluntary self-exclusion from formal financial markets. Voluntary self-exclusion arises when a person is unwilling to participate in the formal financial market —*on the terms that they face*. Transactions costs, risk, and asymmetric information create a spread between the formal market rate of interest that a person is charged for borrowing and the rates she is paid for saving. Furthermore, this gap varies across the population in a systematic manner. The spread tends to be higher for people with less wealth, for the young, and for ethnic minorities. A large gap between high interest charged for borrowing and low interest paid on savings in formal markets makes alternative, informal financial arrangements attractive. These alternatives are not attractive in their own right, they are made attractive in comparison to the terms that formal markets provide.

Informal lending mechanisms compete with formal ones by finding niches that formal financial institutions do not exploit. Friends and family may have more information about a borrower and can encourage repayment by resorting to social suasion that is unavailable to banks. Payday lenders are able to loan small amounts by securing the borrower's next paycheck —something banks do not traditionally do. Loan sharks may threaten violence. Pawnshops accept and store high-value, relatively liquid durable goods as collateral for loans that are a fraction of the expected sale value of these goods. NGO-supported micro finance groups seek to collect information on the borrowers that is different than what is available to commercial banks and may try to build social capital that can be used for social suasion.

Such mechanisms notwithstanding, the spread of informal lending mechanisms among the poor is ultimately constrained by the same difficulties that limit formal financing: fixed transactions costs, a lack of *a priori* information on creditworthiness, and no collateral in the event of default. Most of the world's poor remain financially excluded, awaiting a mechanism that allows them to benefit from pooling their financial resources in the same way that wealthy people do: "*The challenge remains to find ways to deliver small loans and collect small deposits while not sending fees and interest rates through the roof. And if that objective cannot be met, the challenge is to develop a framework for thinking about microfinance as a social tool that might need to rely, to some degree and in some places, on continuing subsidies.*" (Armendáriz, 2010, 318)

4 Lotteries as Financial Instruments for the Poor

In this section I explore using frequent small lotteries with small prizes to bridge the financial gap. I begin by looking at fair lotteries and hypothetical lottery clubs and end by proposing a compensated lottery that self-targets the financially excluded population. This is where I make use of the lottery games and losers distribution developed at the beginning of this paper.

A fair lottery is defined as a lottery where the ticket price is equal to the expected value of the prize. Letting z be the ticket price, Z the lottery prize and p the probability of winning, a lottery is said to be fair when

$$z = E(Lottery) = pZ \tag{11}$$

When viewed in isolation, widespread participation in games of chance with even odds is difficult to reconcile with rational, fully informed, risk-neutral behavior. The reasoning behind proposing fair lotteries as a bridge to financial inclusion is based on how formal markets financially exclude the poor. When compared to financial markets with a very high interest spread, fair lotteries can look like attractive alternative financial instruments. If individual time preference rates fall between the interest rate spread, a risk-neutral, would-be borrower will prefer inter-temporal games with a zero expected present value to formal financial markets. As the interest rate spread widens, more risk-averse borrowers will be attracted. Since lack of collateral, high information costs, short earnings-histories, ethnic and class discrimination all translate into a higher rate spread, it is more likely that the poor, the young, and women and minorities who face discrimination in formal financial institutions will prefer the fair lottery. The attraction is likely to be greatest for recent migrants, ethnic minorities, and for the young, the groups that account disproportionately for the urban poor in developed and developing countries alike.

4.1 Fair Lottery Clubs in an Honest World

I begin by ignoring default risk and transactions costs. Everybody is identical and assumed to be honest and to be able to meet their payment obligations. Consider a financial market in which the interest rate on savings is zero and where person "i" is charged $r_i > \delta$ for borrowing, where δ is his time preference rate. In such a world, a club built around a lottery game can generate significant welfare gains for all but one of its members —and he will be no worse off. A lottery club is formed at time zero when n people agree to play a game that has the following rules. The game will consist of n identical fair lotteries spaced over equal time intervals of duration t, for a total game duration of $T = nt$. Before every lottery, each member contributes $\$z$ to create a total prize of $\$Z = n\$z$. When the first game takes place at time $t = 1$, all member names are placed in a bin and one name is drawn at random. The winner is given the prize of $\$Z$. At time $2t$ the second lottery takes place in a manner that is identical to the first one, except that there are now $(n-1)$ names to draw from, because the first winner's name has been removed (no replacement). At time $3t$ the third lottery is played with $(n-2)$ member names in the bin. This is repeated n times, until all n participants have "won" $\$Z$.

It is easy to show that any risk-neutral person with a positive discount rate will prefer to join a lottery club —*a priori* —to saving in any financial market that has an interest rate split which spans their discount rate. Similarly, given any degree of risk-aversion, there exists a market interest rate spread such that lottery club membership will be preferred. Therefore, if (a) people have a positive discount rate, (b) they are risk-averse, and (c) the formal financial market interest rate split is a decreasing function of wealth, poorer people will self-select into the lottery clubs. Notice also that a weakly risk-averse person has no incentive to join a club unless she is financially constrained. What proportion of the population will choose to join a lottery club depends on the distribution of income, the distribution of risk-aversion, and on the size of the interest rate split that people face in formal financial markets. *A posteriori* the first $(n-1)$ club members to receive a payout will better off having joined in the lottery game than if they had saved at zero interest and the n^{th} player will be indifferent. The lottery game therefore generates a Pareto improvement.

Such social gains will be especially large in markets with high rates on borrowing and non-positive interest rates paid on formal savings accounts —a financial environment that is all too familiar to the poor throughout the world. The break-even constraint implied by fairness rules out private provision. So –in the absence of default risk –it can be argued that fair lottery clubs are public goods.

4.2 Fair, Open Lotteries in a Not so Honest World

Default risk creates a fatal problem for lottery clubs. Once a member has received her prize, she will want to quit. She has no incentive to return to the game and continue making payments. If early winners default, later prizes will be less than $\$Z$, the odds of winning the lottery will no longer be fair, and the club will fall apart. This default risk can be overcome —at a cost.

Consider a fair lottery which pays $\$Z$ with probability p and charges z per ticket, with $z = pZ$, as before. However now allow the lotteries to be open. No club membership is required. Anybody can play and they can play whenever they want to and for as long as they want to. Identical lotteries are held every t intervals of time forever and Z is set to be small –say at some fraction of the local poverty line or at a multiple of the local minimum monthly wage. The probability of winning, p, is large. For instance t and p can be set so that $\frac{t}{p}$ is approximately equal to one year. If the lottery is held every week, then $p = \frac{1}{52}$.

The reasoning for lottery clubs carries through to the open lottery. Risk-neutral people will prefer the open lottery to participation in financial markets if the formal interest rate split spans their time preference rate and if they are financially constrained. Therefore, the poor will self-select into the market for lottery tickets. In fact exactly the same number will want to participate in an open lottery as would join lottery clubs if clubs were offered instead. Also, for any given level of risk aversion, there exists a formal market interest rate spread such that participation in the open lottery will be preferred. But, because a fair open lottery can not generate profits, it will have to be publicly provided. Fairness and private provision are not compatible.

However, —in contrast to the lottery clubs —open fair lotteries will generate Pareto improvements only in the *a priori* sense. The utility of people who wish to play will rise when the lottery is made available and no one's utility will fall —this is revealed by the decisions to buy tickets.

The *ex post* situation is more complicated. If a risk-neutral person with a positive time preference rate plays the lottery every time it is offered for n periods, will he say that he is better off having played? Or will he say that he wishes he had saved the ticket money instead? His answer will depend on whether, when, and how many times he won. If he played every period, for $n = \frac{1}{p}$ times and he won at least once at a time $t^* < n$, he will be better off having played the lottery and will not wish he had saved instead. If he won more than once he will be happy indeed. If he won only once and his win took place in period n he will be indifferent. But if he lost n consecutive times, he will wish he had saved his money instead.

Next, consider a situation where the lottery game is offered and a very large number of people choose to play, i.e. $k \gg n$. When the game concludes, set aside all of the people who won only once. From this group of one-time winners, select one winner from each of the n lotteries in the game. This is now a group of n people with one winner for each of the n lottery dates. The collective outcome for this subset of players is exactly the same as for the perfect world lottery club described earlier. Within this group, resources were pooled and redistributed in a Pareto dominant manner. But, how many similar subsets can be formed out of the set of people who chose to participate in the open lottery?

Once all such groups have been drawn from the full set of players, the remainder will consist of players who won more than once and players who lost every time.[3] Though fair *a priori*, this lottery will result in a transfer from losers to winners almost surely. Those who win at least once will be glad that they played, and those who lose every time will wish that they had saved their money instead. This transfer of welfare among players is at the root of objections to publicly-sponsored lotteries, even if they are fair.

The open lottery generates a potential Pareto improvement overall, but an actual Pareto improvement *only among the subset of winners* $\mathfrak{W}$. The losers, $\mathfrak{L}$, will be strictly worse off *ex post*. How many losers will there be? This proportion is stochastic and is in fact the random variable "$\boldsymbol{l}$" from the losers distribution developed earlier in this paper and described in *Equation* (7). *Figure* 5, below, shows that the proportion of players who lose converges to $\frac{1}{\mathbf{e}}$ —about 37% —for large values of n and lies between 35 and 36% for values of n that would be used in practice[4].

Can these losers be compensated in some way?

5 The Compensated Lottery Game

Consider a compensated open lottery game. It shares many characteristics with the *un*subsidized open lottery described in the previous section.

1. The compensated lottery game consists of a sequence of n lotteries spaced out in equal intervals of duration t,
2. Setting time units equal to one (week, month, year...), the game has a total duration of n.
3. Each lottery pays a prize of Z with probability $\frac{1}{n}$.
4. The game is open:
   (a) anyone can play,
   (b) players can play as many times as they wish, and
   (c) they have no obligation to join a club or to continue playing.
5. Any player who can prove that she has had n consecutive losses will be paid the prize of Z.
6. Every win resets the clock.

When there is no compensation and the ticket is set to $z = pZ$, as before, each lottery in a game has an expected value of zero. So the expected value of the entire game is also zero. This meant that the

[3] Note that the *average* number of wins in this group will be one.
[4] Values in this range are not visible in *Figure* 5 because of scaling problems

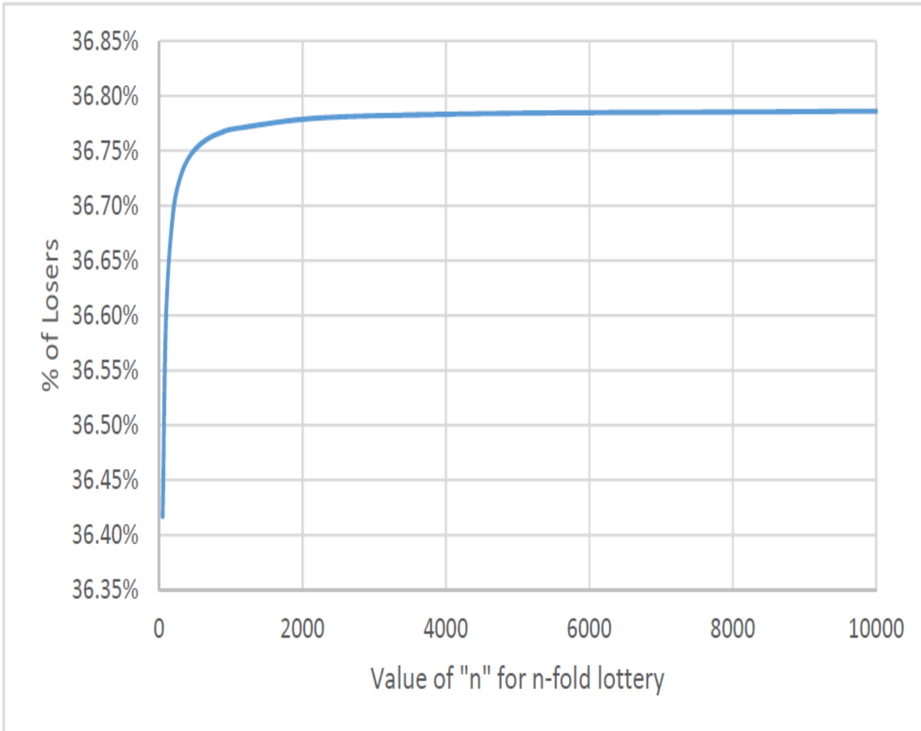

Figure 5: Expected Share of Losers
(As a share of lottery size 'n')

*un*compensated lottery instrument would automatically target people who are financially excluded. It also meant that the uncompensated lottery would be self-financing.

The crucial distinction of the compensated lottery is characteristic 5, the compensation of losers. However this win guarantee causes the compensated lottery game to have a *positive* expected value. The act of compensating losers therefore destroys the ability of the compensated game to automatically target people who are financially excluded. A compensated lottery game with fair pricing for each lottery will therefore crowd out formal financial markets and also be expensive to subsidize. The value of the win guarantee must be priced-in in order to restore automatic targeting —but at what price? The subsidy that is needed will also depend on how many tickets are sold and how that price is set.

The Price of Financial Exclusion In an economic environment in which there is financial exclusion the relevant basis for determining the attractiveness and viability of a new financial instrument is not the positive or negative expected value of the instrument itself, but how it compares to alternative opportunities in the formal financial sector.

Consider a formal financial market where the interest rate for loans charged to person "i", $r_i = r(\mathbf{\Gamma_i})$, is based a vector of personal characteristics $\mathbf{\Gamma_i}$. Consumption technology is given by $C = Y_i + \alpha Z$, where Y_i is exogenous income per period, Z is an indivisible asset, machine, or production technology which yields a return of α per period, with $0 \leq \alpha \leq 1$[5]. If she saves $\frac{Z}{n}$ per period on her own, she has to wait n periods to obtain Z. If she borrows, she obtains Z immediately but has to pay $\frac{r_i(1+r_i)^n}{((1+r_i)^n-1)}Z$ per period —which is more than $\frac{Z}{n}$. Since she will own Z after n periods with both methods, consumption in periods $t > n$ will be identical under both courses of action and she therefore will not involve them in her decision on whether to take out a commercial loan or self-finance. Define person "i" as "financially excluded" if she faces a commercial interest rate $r(\mathbf{\Gamma_i})$ which is so high that she does not participate in the formal market. This requires that the present discounted value of the self-financing option exceed the commercial one:

$$(Y - \frac{Z}{n})\sum_{t=1}^{n} \frac{1}{(1+\delta)^t} > \left(Y + \alpha Z - \frac{r_i(1+r_i)^n}{((1+r_i)^n - 1)} Z\right) \sum_{t=1}^{n} \frac{1}{(1+\delta)^t}, \tag{12}$$

which simplifies to

$$\left(\frac{r_i(1+r_i)^n}{(1+r_i)^n - 1}\right) - \left(\frac{1}{n}\right) > \alpha; \tag{13}$$

financial exclusion exists when the additional cost of a commercial loan in each period exceeds the return on Z.

This means that the price of a compensated lottery ticket can be honed to self-target the financially excluded population, without drawing resources from existing formal financial markets. It creates a new financial market among the poor by pooling their resources.

<u>THEOREM 3:</u> The financial exclusion ticket price

Designate r_e as the commercial interest rate charged on borrowing such that if $r(\mathbf{\Gamma_i}) \geq r_e$, person "$i$" is considered to be financially excluded. With consumption technology $C = Y + \alpha Z$ and discount rate δ, the compensated lottery ticket price can be set to a value $z(r_e)$, such that only the financially excluded population will want to participate. The "financial exclusion ticket price" $z(r_e)$ is given by

$$z(r_e) = Z\left(\sum_{t=1}^{n} \frac{\frac{r_e(1+r_e)^n}{(1+r_e)^n - 1} - \alpha[1 - p(1-p)^{(t-1)}]}{(1+\delta)^t \sum_{t=1}^{n} \frac{(1-p)^{(t-1)}}{(1+\delta)^t}}\right) \tag{14}$$

[5] A similar framework is employed in the economics of poverty traps (see Carter and Barrett, 2006 and Barrett and Carter, 2013).

Proof. The compensated lottery is preferred to a formal sector loan if

$$\sum_{t=1}^{n} \frac{Y + (p\alpha Z - z)(1-p)^{t-1}}{(1+\delta)^t} > \sum_{t=1}^{n} \frac{Y + Z\left[\alpha - \left(\frac{r(1+r)^n}{(1+r)-1}\right)\right]}{(1+\delta)^t},$$

which reduces to

$$\sum_{t=1}^{n} \frac{\left(\frac{r(1+r)^n}{(1+r)^n-1}\right)Z - z(1-p)^{t-1}}{(1+\delta)^t} > \sum_{t=1}^{n} \frac{\alpha Z\left[1 - p(1-p)^{t-1}\right]}{(1+\delta)^{t-1}}.$$

In the formal financial arrangement, a loan makes Z available with certainty at the outset. The date when it will become available under the compensated lottery is stochastic. So the compensated lottery is preferred to formal finance when the difference between formal financing costs and expected spending on lottery tickets is greater than the expected difference in services from Z that will be forgone under lottery game financing over the duration of the financing period N. The result in equation (14) is obtained when the two terms in this inequality are set equal and solved for the lottery ticket price $z(r_e)$. Note that the capital recovery factor —the term involving r— is strictly increasing in r, so r_e is the lower bound of the set of commercial interest rates r_i for which the compensated lottery is be preferred to a commercial loan. ■

When an open lottery game is not compensated, it is not clear what will happen after a player wins. He may choose to stop playing, or he might decide to play again. Uncompensated lottery payoffs have no memory —they are strictly independent from the history of wins. But the compensated game's rule —the rule which requires n sequential losses for compensation— links the expected lottery payoffs to a player's win history. This linkage removes any behavioral ambiguity. Since the game only lasts for n periods, anyone who wins at a time $t < n$ is disqualified for compensation. This means that once a person has won, the expected value of the remaining set of lotteries in the game becomes negative. This is because the price of each lottery is set at $z_e > pZ$. The personal situation of a winner may signify that Z is not enough, that he wants to continue playing. However if he does want to continue playing, he will prefere to start a new game —so that losses qualify for compensation again — rather than continue participating in the game where he has already won and therefore does not qualify for compensation.

The subsidy thus emerges from the scenario where every player who wins "defaults" after their first win. Knowing this —together with the financial exclusion ticket price and the expected number of n-times losers— makes it possible to calculate the expected subsidy. The difference between ticket sale revenues and prizes paid will depend on the ticket price z_e, the total number of players "k", the duration of each game "n", the lottery prize "Z", the probability of winning any given lottery "p", on "$\boldsymbol{l}$", the number of n-fold losers who will have to be compensated, and on "α", the per-period return on Z.

<u>Theorem 4:</u> The Subsidy for a Compensated Lottery

Consider an n-fold lottery game that is compensated in the sense that any player who demonstrates n consecutive losses is awarded the lottery prize Z. This game will require a subsidy s that is given by

$$s = ZK\left\{\sum_{t=1}^{n} \frac{\frac{r_e(1+r_e)}{(1+r_e)-1} - \left[1-p(1-p)^{(t-1)}\right]\alpha}{(1+\delta)^t} - \sum_{t=1}^{n} \frac{(1-p)^{(t-1)}}{(1+\delta)^t} - \frac{l}{(1+\delta)^n}\right\} \tag{15}$$

Proof. The n-lottery game begins with K players purchasing tickets for the round 1 lottery. Of these pK win and drop out. This generates revenues of $z(r_e)K$ and pKZ in prize costs. The $K(1-p)$ players left for the second round generate revenues of $z(r_e)(1-p)K$ and win prizes worth $p(1-p)KZ$. This continues through period n, when $(1-p)^{(n-1)}K$ players generate revenues of $z(r_e)(1-p)^{(n-1)}K$ and take prizes worth $p(1-p)^{(n-1)}KZ$. The difference between revenues and expenses over the entire n periods is therefore

$$z(r_e)K\sum_{t=1}^{n} \frac{(1-p)^{(t-1)}}{(1+\delta)^t} - pZK\sum_{t=1}^{n} \frac{(1-p)^{(t-1)}}{(1+\delta)^t} - \frac{\boldsymbol{l}KZ}{(1+\delta)^n}$$

The first term is revenues. The second term is expenses for prizes paid. The third term, that involves $\boldsymbol{l}$, is the expected spending on compensation to loosers in the final period. All three terms are expressed in present value in order to make them conformable for addition. Substituting for $z(r_e)$ from equation (14) and factoring out ZK yields equation (15), the desired result. ■

Table 2: **Compensated Lottery Game Simulation (1,000 Players, $1,200 Prize)**

Game Duration (n=months)	Ticket Price z_e	Subsidy per '000 Players	Total Savings Generated	# People Receive Prize Before n
6	\$295.15	\$12,332	\$1,200,000	665
12	\$147.28	\$31,888	\$1,200,000	648
18	\$96.27	\$50,125	\$1,200,000	643
24	\$70.48	\$67,223	\$1,200,000	640
60	\$23.90	\$149,511	\$1,200,000	635

Table 2, above, illustrates how the compensated lottery would work for a monthly lottery that lasts for n months and pays a $\$1,200$ prize with probability $\frac{1}{n}$. Simulations in the table assume $\delta = \alpha = r_e = 10\%$ per year. Values of n shown range from 6 to 60 months. Since the prize is held fixed in these simulations, the ticket price $z(10\%, n)$ ranges from \$23.90 for a monthly lottery that lasts for 5 years to \$295.15 for a 6-month lottery with the same \$1,200 prize. In six months, a compensated lottery will generate $\$1,200,000$ in savings among every thousand financially excluded participants in exchange for a subsidy of $\$12,332$ —about 1%.

Seen another way, every subsidy dollar spent on the lottery will generate \$97.31 in savings among the poor! Also, 665 people out of every thousand who choose to play the lottery will receive their $1,200$ earlier than if they had saved the money on their own. This is not a quantity to be ignored and may prove to be the largest positive net impact of a subsidized, compensated lottery. Table 2 shows that these values range monotonically with the lottery duration and prop .

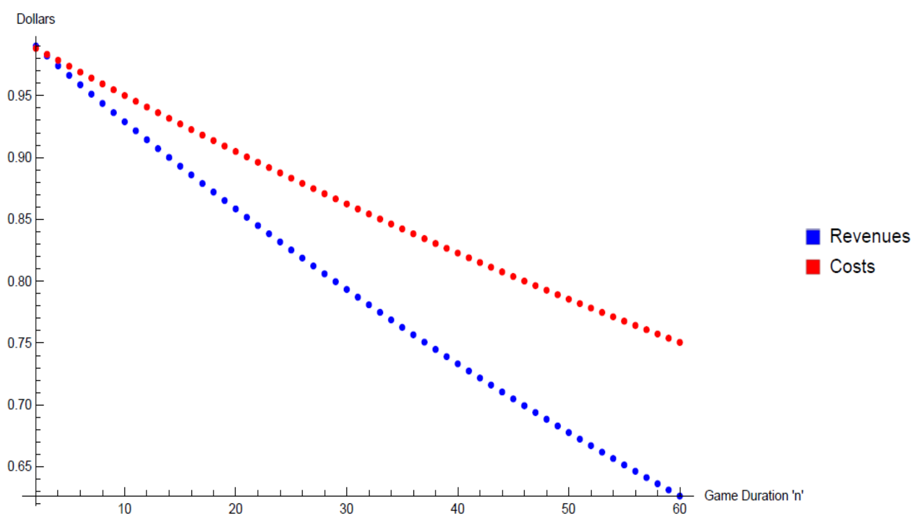

Figure 6: Revenues & Costs per Dollar of Prize Money (10% Yearly Discount Rate, Time in Months).

The only way to reduce monthly lottery ticket price —while keeping the prize constant and keeping the lottery attractive to the same group of financially excluded people— is to increase n. This increases its duration and reduces the probability of winning any one lottery. Though counterintuitive, it turns out that increasing n makes the lottery *more* expensive to subsidize —even when the prize money and total number of players is held constant. Figure 6 shows that costs fall more slowly than revenues, so subsidies must increase. This need for a higher subsidy at bigger n values is also evident in the Table 2. As game duration rises, both revenues and costs fall. Discounting reduces both costs and revenues when the game is stretched out over a longer time. Also, stretching out the duration of the game reduces the probability that a player will win any given lottery and —for a fixed number of players— this reduces costs because it reduces prize payouts per period. The longer duration and lower win probability also reduces willingness to pay for the chance to win a given prize and therefore also lowers the feasible ticket price —and so reduces revenues. But as n rises, the major difference in the paths of cost and revenues arises from the fact that there are more people who need to be compensated. This is because $\hat{l}$ rises with n and it rises most steeply for the lower values of n over which most practical lotteries will be organized. The rising proportion of losers only affects costs, not revenues. Like an externality, this higher cost is not internalized by individual players, and so it must be born by the group as a whole.

Desirable values of n for the compensated lottery are likely to be quite small —weekly or monthly drawings for one or two years at most. Administrative costs are likely to be low since there is no roll for collateral, collections, information asymmetry, or default risk. The biggest advantage of compensated lotteries lays in the fact that there is no role for gender, racial, and ethnic discrimination in lottery-based financing as long as anyone can buy a ticket.

5.1 Lotteries Compared to Microfinance

A major consideration in using compensated lotteries as a way to reach the financially disenfranchised population is likelty to be its cost. The lottery authority could consider controling the amount of the subsidy by limiting the number of tickets sold. It can also do so by manipulating p, or by requiring proof of $n + \varepsilon$ consecutive losses before a prize is awarded to a loser, with $\varepsilon \geq 1$. However, it should be understood that —while values of $\varepsilon > 0$ or $p < \frac{1}{n}$ lower the expected value of the subsidy— they do so by making the lottery game more expensive and so partially defeat its purpose of supporting the population that is financially disenfranchised.

Alternative microfinance mechanisms have mushroomed over the past few decades. If compensated lotteries are going to be considered, their cost must be compared to the cost of these other efforts to provide financial markets to the poor. How would a subsidized lottery compare to the typical terms of a micro-loan?

Microfinance has undoubtedly created a source of financing for a very large number of people who can not participate in formal financial markets —or chose not to participate at the commercial terms they face. By the end of 2013 there were 3,098 microfinance institutions lending to over 211 million borrowers (Microcredit Summit Campaign, 2016). Yet these organizations are typically dependent on overt and hidden subsidies, charge high interest rates, and are plagued by default. Lafourcade et al. (2006) analyzed 163 microfinance organizations in 11 African countries and found that only 47% earned positive returns –returns are typically based on calculations that count subsidies as revenue. So "*the reality is that much of the microfinance movement continues to take advantage of subsidies*" (Armendáriz and Morduch, 2010, p. 318).

In a very thorough recent study Cull, Morduch, and Demirgüç-Kunt (2018) analyze proprietary data on 1,335 microfinance institutions that serve over 80 million borrowers globally. They find that "*subsidy remains pervasive*," averages about 13 cents per dollar spent on loans, but that this rate is highly skewed. Despite having existed for about half a century now, apparently no way has yet been found to fully integrate microfinance for the poor into formal financial markets. Microfinance markets remain segmented, with NGO's accounting for the bulk of the smallest loans, formal banks accounting most larger "microloans" —median loans 7 times larger than the NGO's— and non-bank financial institutions (NBFIs) covering the middle ground. They also find that lending costs are very similar across NGO's, NBFIs, and formal banks and that fixed costs dominate the cost of lending. Lending costs are therefore relatively 'flat' with respect to loan size, with the result that cost per loan is higher for the NGO's because they specialize in granting smaller loans. The most troubling finding of Cull, Morduch, and Demirgüç-Kunt (2018) is that —-as a result of higher costs —"*the poorest customers in the microcredit sector pay the highest interest rates.*" The average rate charged to the poorest customers ranges from 30 to 40%.

Banerjee, Karlan, and Zinman (2015) review some of the first causal studies of microfinance programs. They review 7 programs in Bosnia, Ethiopia, India, Mexico, Mongolia, and Morocco and find evidence of only modest impact. Though explicitly seeking to address financial exclusion, all 7 programs placed severe restrictions on borrower eligibility and charged high —sometimes exorbitant— interest rates. These include

- Bosnia: limited to microentrepreneurs posessing sufficient collateral, repayment capacity, credit worthiness, business capacity, and credit history; 22% APR interest.
- Ethiopia: limited to microentrepreneurs, joint group liability, proof of poverty status, posessing sufficient collateral, and evidence of a viable business plan; 12% APR interest.
- India: limited to females, requires joint group liability, home ownership, local residence for at least 3 years; 24% APR interest.
- Mexico: limited to female microentrepreneurs with proof of business ownership or significant economic activity, group liability; 110% APR interest.
- Mongolia: limeted to female microentrepreneurs, group liability; 26.8% APR interest.
- Morocco: limited to microentrepreneurs with proof of economic activity for at least 12 months, group liability; 14.5% APR interest.

Banerjee, Karlan, and Zinman (2015) report low take-up rates in the 5 studies that used randomized program placement. They also find some evidence that pre-microcredit formal and informal financing is crowded out by these programs. Both the low take-up and crowding are probably caused by the eligibility restrictions.

In comparison to these experiences the compensated lottery has no eligibility restrictions. It requires no proof of collateral or experience, age, ethnicity, or gender. The required subsidy is explicit and low in comparison to the subsidies of most known microcredit organizations. The return on assets that is required to make the compensated lottery an attractive financial instrument can be set to be *much* lower than what is currently the practice in microlending.

6 Conclusions

This paper has developed a new probability distribution, the "Losers Distribution"' and demonstrated its application to the problem of financial exclusion.

Can open lotteries do a better job of extending financial inclusion to the poor? The arguments developed here suggest that they may be able to. They self-target the poor, eliminate the need for screening of any sort, and create a Pareto improvement in the ex-ante sense that those who choose to participate are by definition people who prefer a chance at obtaining a lump sum of cash early to the certainty of having to wait for it.

However, even fair open lotteries randomly redistribute resources. They do not generate Pareto improvements in the *ex post* sense and the pre-lottery distribution of income first order stochastically dominates the post-lottery distribution and therefore will worsen poverty as defined by a wide class of poverty measures (Atkinson, 1987).

An open lottery which compensates losers will restore ex-post Pareto dominance defined over the set of lottery participants. Subsidies per dollar of savings created are estimated to lie in the range of 1% for 6-months to less than 12.5% for 5-year financing appear to be well within the bounds of the subsidies that are currently sustaining microcredit institutions world-wide. At the same time, compensated lotteries should do a much better job of reaching the poorest, most financially excluded segments of the population. When compared to the microcredit initiatives that have been tested over the past fifty years, the administrative costs of a compensated lottery would likely be far lower, and subsidies far more transparent. Take up rates should be much higher.

Finally, the transition from a pre to a post-compensated lottery national income distribution satisfies the Pigou-Dalton transfer principle if lottery subsidies are financed with revenues raised from an income tax with a zero rate bracket for people below the poverty line and only the poor self-select into the lottery.

Compensated lotteries are unlikely to replace microfinance and will certainly not replace formal banking, —they are not intended to. There is a special place for them in financial markets as a the missing link to the financially excluded poor. Formal financial markets take the pool of money from the more patient subset of formal market participants —the “savers”— and turn them into the liabilities of another, less patient group of “borrowers”. Lotteries similarly take a fixed pool of savings from one subset of players and distribute them among a group of “winners”. After an initial burst from pent-up demand, new financial needs will arise continuously —just as they do in formal financial markets— creating a more or less steady flow of demand for lottery tickets. A major difference between these financial instruments will be who chooses to uses them. If the subsidy is set appropriately, compensated lotteries will not crowd out formal financial markets, they will mobilize the financial resources of the poor and bring in a new set of customers who have no better alternative.

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References

Andrews, George E. 1984. *The theory of partitions*, New York, Cambridge Mathematical Library.

Armendáriz de Aghion, B., and Morduch, J. 2010. *The Economics of Microfinance*, 2nd edition, Cambridge, Mass: MIT Press.

Atkinson, A.B. 1987. “On the measurement of poverty.” *Econometrica*, 55 (4): 749-764.

Banerjee, Abhijit, Dean Karlan, and Jonathan Zinman. 2015. “Six Randomized Evaluations of Microcredit: Introduction and Further Steps.” *American Economic Journal: Applied Economics* 7 (1): 1–21.

Christopher B. Barrett and Michael R. Carter. 2013. “The Economics of Poverty Traps and Persistent Poverty: Empirical and Policy Implications.” *The Journal of Development Studies* 49 7:

976 – 990. DOI: 10.1080/00220388.2013.785527

Campbell, John Y., Howell E. Jackson, Brigitte C. Madrian, and Peter Tufano. 2011. "Consumer Financial Protection." *Journal of Economic Perspectives* 25 (1): 91-114.

Carter, Michael R. and Christopher B. Barrett. 2006. "The economics of poverty traps and persistent poverty: An asset-based approach." *The Journal of Development Studies* 42 2: 178 – 199. DOI: 10.1080/00220380500405261

Chetty, Raj. 2006. "A New Method of Estimating Risk Aversion." *American Economic Review,* 96 (5): 1821-1834. DOI: 10.1257/aer.96.5.1821

BBC News. October 4, 2018. Italian 'Robin Hood' banker sentenced over 1m pilfered for the poor. Retrieved from https://www.bbc.com/news/world-europe-45745082

, Rebel A. and John D. Wolken. (1995),]. Financial Services Used by Small Businesses: Evidence from the 1993 National Survey of Small Business Finances. Federal Reserve Bulletin, 81:July, 801-817.

Consumer Financial Protection Bureau. 2013. *Financial Literacy Annual Report.*

Cull, Robert, Jonathan Morduch, and Asli Demirgüç-Kunt. 2018. "The Microfinance Business Model: Enduring Subsidy and Modest Profit." *World Bank Economic Review. forthcoming.*

Faris, W. G. 2011. "Multinomial coefficients: Notes from Math 447-547 lectures", University of Arizona. Retrieved from http://math.arizona.edu/ faris/combinatoricsweb/multinomial.pdf

Feller, W. 1957. *An introduction to probability theory and its applications.* New York, Wiley.

Guillen, M. and Tschoegl, A. 2002. "Banking on Gambling: Banks and Lottery-Linked Deposit Accounts," *Journal of Financial Services Research* Vol. 21, No. 3.

Kao, R. C. and L. H. Zetterberg 1957. "An identity for the sum of multinomial coefficients", *The American Mathematical Monthly* 64:2, 96-100.

Lafourcade, Anne-Lucie, Jennifer Isern, Patricia Mwangi, and Matthew Brown. 2006. "Overview of the Outreach and Financial Performance of Microfinance Institutions in Africa." *Microbanking Bulletin,* 12, April, 3-14.

Lawrence, Edward C., and Gregory Elliehausen. 2008. "A Comparative Analysis of Payday Loan Customers." *Contemporary Economic Policy,* 262.:299–316.

Microcredit Summit Campaign February, 2016. *State of the campaign report 2015.* Retrieved from https://stateofthecampaign.org/data-reported/

Morduch, Jonathan. 1999. "The Microfinance Promise." *Journal of Economic Literature* 37 (4): 1569-1614.

Polgreen, Lydia, and Vikas Bajaj. 2010. "India Microcredit Faces Collapse From Defaults." *New York Times,* November 17. https://www.nytimes.com/2010/11/18/world/asia/18micro.html

Zeelenberg, M., and Pieters, R. 2004. Consequences of regret aversion in real life: The case of the Dutch postcode lottery. *Organizational Behavior and Human Decision Processes* 93, 155-168.

7 APPENDIX

TABLE A1									
1	2	3	4	5	6	7	8	9	10
# of winners "w"	Unique partitions of the number 9 or "*R* - Vectors"	p(9,w): # of partitions of 9 into w parts	# of ways in which this Win Vector *Exactly can occur (multinomial coefficient of* R-*vector)*	Combinations of "w" winners with this vector	$Q(9,X_i)$: Prob. Of This Win Vector *for any Winner (Col.4 x Col.5)*	$\sum Q(9, w_i)$ Q-Triangle Entries	Binomial Coefficients: Pascal's triangle entries	$L(9,w_j)$: Prob. Of This Win Vector *for any Winner (Col.6 x Col.8)*	$\sum L(9, w_j)$ = Col. 7xCol. 8 L-Triangle Entries
9	(1,1,1,1,1,1,1,1,1)	1	362,880	1	362,880	362,880	1	362,880	362,880
8	(2,1,1,1,1,1,1,1)	1	181,440	8	1,451,520	1,451,520	9	13,063,680	13,063,680
7	(3,1,1,1,1,1,1)	2	60,480	7	423,360	2,328,480	36	15,240,960	83,825,280
	(2,2,1,1,1,1,1)		90,720	21	1,905,120		36	68,584,320	
6	(4,1,1,1,1,1)	3	15,120	6	90,720	1,905,120	84	7,620,480	160,030,080
	(3,2,1,1,1,1)		30,240	30	907,200		84	76,204,800	
	(2,2,2,1,1,1)		45,360	20	907,200		84	76,204,800	
5	(5,1,1,1,1)	5	3,024	5	15,120	834,120	126	1,905,120	105,099,120
	(4,2,1,1,1)		7,560	20	151,200		126	19,051,200	
	(3,3,1,1,1)		10,080	10	100,800		126	12,700,800	
	(3,2,2,1,1)		15,120	30	453,600		126	57,153,600	
	(2,2,2,2,1)		22,680	5	113,400		126	14,288,400	
4	(6,1,1,1)	6	504	4	2,016	186,480	126	254,016	23,496,480
	(5,2,1,1)		1,512	12	18,144		126	2,286,144	
	(4,3,1,1)		2,520	12	30,240		126	3,810,240	
	(4,2,2,1)		3,780	12	45,360		126	5,715,360	
	(3,3,2,1)		5,040	12	60,480		126	7,620,480	
	(3,2,2,2)		7,560	4	30,240		126	3,810,240	
3	(7,1,1)	7	72	3	216	18,150	84	18,144	1,524,600
	(6,2,1)		252	6	1,512		84	127,008	
	(5,3,1)		504	6	3,024		84	254,016	
	(5,2,2)		756	3	2,268		84	190,512	
	(4,4,1)		630	3	1,890		84	158,760	
	(4,3,2)		1,260	6	7,560		84	635,040	
	(3,3,3)		1,680	1	1,680		84	141,120	
2	(8,1)	4	9	2	18	510	36	648	18,360
	(7,2)		36	2	72		36	2,592	
	(6,3)		84	2	168		36	6,048	
	(5,4)		126	2	252		36	9,072	
1	(9)	1	1	1	1	1	9	9	9
Column Sum		30	871,030	256	7,087,261	7,087,261		387,420,489	387,420,489

TABLE A2: *S(n,w):* **Stirling Numbers of the second kind (matrix $\mathcal{M}_2$)**

n↓ / w→	1	2	3	4	5	6	7	8	9	10	Row sums: Bell Numbers (Series A000110)
1	1										1
2	1	1									2
3	1	3	1								5
4	1	7	6	1							15
5	1	15	25	10	1						52
6	1	31	90	65	15	1					203
7	1	63	301	350	140	21	1				877
8	1	127	966	1701	1050	266	28	1			4140
9	1	255	3025	7770	6951	2646	462	36	1		21147
10	1	511	9330	34105	42525	22827	5880	750	45	1	115975

TABLE A3: **Factorials: w! (matrix $\mathcal{M}_3$)**

n↓ / w→	1	2	3	4	5	6	7	8	9	10	Row sums: Σw!
1	1										1
2	1	2									3
3	1	2	6								9
4	1	2	6	24							33
5	1	2	6	24	120						153
6	1	2	6	24	120	720					873
7	1	2	6	24	120	720	5040				5913
8	1	2	6	24	120	720	5040	40320			46233
9	1	2	6	24	120	720	5040	40320	362880		409113
10	1	2	6	24	120	720	5040	40320	362880	3628800	4037913

TABLE A4: Q(n,w)=w!S(n,w): **Q-Triangle of win frequencies with no Losers (matrix $\mathcal{M}_4$)**

n↓ / w→	1	2	3	4	5	6	7	8	9	10	Row sums: Fubini Numbers (Series A000670)
1	1										1
2	1	2									3
3	1	6	6								13
4	1	14	36	24							75
5	1	30	150	240	120						541
6	1	62	540	1560	1800	720					4683
7	1	126	1806	8400	16800	15120	5040				47293
8	1	254	5796	40824	126000	191520	141120	40320			545835
9	1	510	18150	186480	834120	1905120	2328480	1451520	362880		7087261
10	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800	102247563

TABLE A5: **Pascal's Triangle of binomial coefficients (zero column excluded) (matrix $\mathcal{M}_5$)**

n↓ / w→	1	2	3	4	5	6	7	8	9	10	Row sums: Gausian Binomial Coefficients
1	1										1
2	2	1									3
3	3	3	1								7
4	4	6	4	1							15
5	5	10	10	5	1						31
6	6	15	20	15	6	1					63
7	7	21	35	35	21	7	1				127
8	8	28	56	70	56	28	8	1			255
9	9	36	84	126	126	84	36	9	1		511
10	10	45	120	210	252	210	120	45	10	1	1023

TABLE A6: L(n,w), **Losers Triangle, Hadamard product: $\mathcal{M}_6=\mathcal{M}_4\circ\circ\mathcal{M}_5$**

n↓ / w→	1	2	3	4	5	6	7	8	9	10	Row sums: k^n, the Sample Space
1	1										1
2	2	2									4
3	3	18	6								27
4	4	84	144	24							256
5	5	300	1500	1200	120						3125
6	6	930	10800	23400	10800	720					46656
7	7	2646	63210	294000	352800	105840	5040				823543
8	8	7112	324576	2857680	7056000	5362560	1128960	40320			16777216
9	9	18360	1524600	23496480	105099120	160030080	83825280	13063680	362880		387420489
10	10	45990	6717600	171889200	1285956000	3451442400	3556224000	1360800000	163296000	3628800	10000000000