EasyChair Preprint

№ 175

Computation of rational parameter dependent Lyapunov functions for LPV systems

Péter Polcz, Balázs Á. Kulcsár and Gábor Szederkényi

EasyChair preprints are intended for rapid dissemination of research results and are integrated with the rest of EasyChair.

May 27, 2018

Computation of rational parameter dependent Lyapunov functions for LPV systems ⋆

Peter Polcz * **Balázs Kulcsár** ** **Gábor Szederkényi** *,***

* *Faculty of Information Technology and Bionics, Pázmány Péter Catholic University, Práter 50/a, H-1083 Budapest, Hungary (e-mail: polcz.peter,szederkenyi@itk.ppke.hu).*
** *Automatic control research group, Chalmers University of Technology, Hörsalsvägen 9-11, 412 58 Göteborg, Sweden (e-mail: kulcsar@chalmers.se)*
*** *Systems and Control Laboratory, Institute for Computer Science and Control (MTA SZTAKI), Hungarian Academy of Sciences, Kende 13-17, H-1111 Budapest, Hungary*

Abstract:
In this contribution, we present a computational method for the global stability analysis of linear parameter varying systems under rational parameter dependence. Using the linear fractional representation (LFR) of the system equation, we generate a set of rational basis functions, which will give the structure for the parameterized rational Lyapunov function. Based on the earlier results of Trofino and Dezuo (2013), affine parameter dependent linear matrix inequalities (LMIs) are formulated to ensure the Lyapunov conditions and hence asymptotic stability.

Keywords: linear parameter varying systems, global stabily, Lyapunov functions, LMI

1. INTRODUCTION

The computation of Lyapunov functions for dynamical system models caught a large amount of attention in the past few decades e.g. Vannelli and Vidyasagar (1985). Effective methods have been elaborated to uncertain linear systems, e.g. Khargonekar et al. (1990).

The birth of the theory of linear parameter varying (LPV) systems e.g. Wu (1995); Wu and Dong (2006) inspired researchers to use parameter dependent Lyapunov functions (PDLFs) to test robust stability conditions from different angles e.g. Seiler et al. (2009). Linear stability analysis methods (and some of the nonlinear ones) propose to use numerically effective tools, such as LMIs (Scherer and Weiland (2000)) to approach robust stability via convex optimization. Structuring PDLFs is a common way, e.g. affine (Cox et al. (2018)), polytopic (Trofino and Dezuo (2013)). Note that some methods have been approaching to address robust stability by means of sum-of squares (SOS) method to reduce the numerical conservativeness of convex relaxations e.g. Wu and Prajna (2005). This latter however does not scale very well to handle larger dimensional problems.

In Iwasaki and Shibata (2001), quadratic separators have been proposed to split the analysis problem into lower dimensional and separated conditions, i.e. nominal and uncertain parts. The engine behind the separation has been triggered by Finsler's lemma. In this vein and based on frequency domain considerations, they proposed LMI conditions for the global stability of linear-time invariant systems with an uncertain (possibly time-dependent) algebraic constraint. These results were extended for the global stability analysis of LPV systems in the linear fractional representation using rational Lyapunov functions (L.f.s) containing uncertain parameters.

Selecting a parameterized uncertain rational L.f. candidate, Trofino and Dezuo (2013) used Finsler's lemma and affine annihilators to obtain polytopic parameter dependent LMIs ensuring the Lyapunov conditions. In case of LPV systems, the structure of the L.f. candidate is the same in both references and the system representation is given in a similar form with an algebraic constraint. The technique introduced by Trofino and Dezuo (2013) and improved by Polcz et al. (2017, 2018) is capable to handle any locally asymptotically stable nonlinear system in the linear fractional representation.

In this work, we apply the dual stability conditions to reach global stability analysis of LPV systems under rational parameter dependence. The main idea is to remove the possible state dependence from the generated parameter dependent LMI conditions, therefore, the feasibility of these new LMIs ensure global stability inside the initially given parameter domain. The proposed numerically effective approach has been tested on a second order LPV model of computational interest, furthermore, it is compared to the method of Iwasaki and Shibata (2001).

⋆ This research has been partially supported by the European Union, co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017-00002). The research was also supported by Pázmány Péter Catholic University through the project KAP17-61005-1.1-ITK.

2. PROBLEM FORMULATION

We start from LPV systems of the form

$$\dot{x} = F(\delta)\, x, \quad F_{ij}(\delta) = \frac{p_i(\delta)}{q_i(\delta)}, \quad i, j = 1, ..., n \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\delta \in \mathbb{R}^d$ are the possibly time dependent parameters, $p_i(\delta)$ and $q_i(\delta)$ are polynomials of δ. We assume that the uncertain parameter vector and its rate are bounded and there exist two bounded polytopes $\mathcal{D}, \check{\mathcal{D}} \subset \mathbb{R}^d$ such that $\delta \in \mathcal{D}$ and $\dot{\delta} \in \check{\mathcal{D}}$.

System (1) can be given in the linear fractional representation, namely

$$\dot{x} = Ax + B\pi, \tag{2a}$$

$$y = Cx + D\pi, \tag{2b}$$

$$\pi = \Delta(\delta)y. \tag{2c}$$

where A, B, C, D are constant matrices of appropriate dimensions, $\Delta(\delta)$ is a diagonal operator. Henceforth, the arguments of Δ will be suppressed. Vectors π, $y \in \mathbb{R}^p$ constitute the input and output, respectively, for the linear time-invariant system (2a-b). The feedback (2c) represents the uncertain part of the dynamics. Eliminating variable y from (2b-c), one can obtain the explicit formula for vector $\pi = \pi(x, \delta) = (I - \Delta D)^{-1}\Delta C x$.

The Lyapunov function candidate for system (1) in representation (2) is searched for in a general quadratic form:

$$V(x, \delta) = \pi_b^T P \pi_b, \text{ where } \pi_b = \left(\begin{smallmatrix} x \\ \pi \end{smallmatrix}\right), \tag{3}$$

furthermore, P is a symmetric not necessarily positive definite matrix of the free decision variables of the LMI problem (i.e. these variables should be chosen such that the LMIs are satisfied). The time derivative of the L.f. considering the dynamics (1) can be written in a similar form as the L.f. itself, namely:

$$\dot{V}(x, \delta, \dot{\delta}) = \frac{\partial V}{\partial x} F(\delta)x + \frac{\partial V}{\partial \delta}\dot{\delta} = \pi_a^T R \pi_a, \tag{4}$$

where π_a is a vector of rational functions of x, δ and $\dot{\delta}$ and $R = R(P)$ is a matrix containing free decision variables. To prove global asymptotic stability of the equilibrium points $x^* = 0$ for (1), function $V(x, \delta)$ should satisfy the well-know Lyapunov conditions, namely $V(x, \delta)$ should be positive definite and its time derivative (4) should be negative definite for all $(\delta, \dot{\delta}) \in \mathcal{D} \times \check{\mathcal{D}}$ and for all $x \in \mathbb{R}^n$.

3. MAIN RESULTS

In this section, we propose a straightforward method for global L.f. construction for LPV systems based on our previous results in Polcz et al. (2017, 2018).

Theorem 1. Consider a dynamical system in the linear fractional representation (2), and a function (3), with its time derivative (4). Let $N_b(\delta)$ and $N_a(\delta, \dot{\delta})$ be two affine matrices (called annihilators) such that

$$N_b(\delta)\pi_b = 0 \text{ and } N_a(\delta, \dot{\delta})\pi_a = 0, \; \forall(\delta, \dot{\delta}) \in \mathcal{D} \times \check{\mathcal{D}}. \tag{5}$$

According to Finsler's lemma (Trofino and Dezuo (2013)), $V(x, \delta)$ is a Lyapunov function for system (2) if the following parameter dependent LMI conditions are satisfied:

$$\begin{aligned} &P + L_b N_b(\delta) + N_b^T(\delta) L_b^T \succ 0, && \forall \delta \in \mathcal{D}, \\ &R + L_a N_a(\delta, \dot{\delta}) + N_a^T(\delta, \dot{\delta}) L_a^T \prec 0, && \forall(\delta, \dot{\delta}) \in \mathcal{D} \times \check{\mathcal{D}}. \end{aligned} \tag{6}$$

To rephrase, the feasibility of LMIs (6) guarantee global stability for (1). Due to the fact that these LMIs are affine expression of the uncertain parameters and their derivatives which belong to bounded polytopes, it is enough to check their feasibility only in the corner points of the polytopes $\mathcal{D}$ and $\check{\mathcal{D}}$.

We shall note that in general, the size of the second LMI condition is significantly larger than that of the first LMI.

3.1 Global stability of a second order LPV model

In this section, we demonstrate the operation of the proposed method. To model and solve LMI problems we used YALMIP (Löfberg (2004)) with SeDuMi solver (Sturm (1999)) in Matlab environment. The computations were processed on a 4th generation Intel Core i7 processor.

Consider a second order LPV system (1) with

$$\mathcal{A}(\delta) = \begin{pmatrix} -\dfrac{\delta}{\delta - 5} - 2 & \delta + 1 \\ 0.2 & \dfrac{3}{\delta - 5} \end{pmatrix}. \tag{7}$$

We assume that the domain of δ and its rate $\dot{\delta}$ are normalized, namely, $\mathcal{D} = \check{\mathcal{D}} = [-1, 1]$.

In order to obtain a simplified LFR model for (7), we used the symbolic LFR technique presented by Polcz et al. (2018), which results in the following (linearly independent) functions to be considered in the L.f. (3):

$$\pi^T = \begin{pmatrix} \dfrac{5\delta x_1}{\delta - 5} & \delta x_2 & \dfrac{\delta x_2}{\delta - 5} \end{pmatrix}. \tag{8}$$

Solving the LMI feasibility problem (6), we obtained the following value for the Lyapunov matrix P:

$$P = \begin{pmatrix} 1.7 & 1.08 & 0.0124 & 1.91 & 4.4 \\ 1.08 & 6.38 & 0.263 & -0.147 & -2.19 \\ 0.0124 & 0.263 & 0.133 & 0.0151 & -0.186 \\ 1.91 & -0.147 & 0.0151 & 1.48 & 0.357 \\ 4.4 & -2.19 & -0.186 & 0.357 & 3.62 \end{pmatrix}. \tag{9}$$

The 3D plot and the level curves of the L.f. $V(x_1, x_2, \delta)$ are illustrated in Figure 1 for five different values of the uncertain parameter. The number and size of the LMIs are presented in Table 1 alongside with the number of free decision variables of the optimization model.

We compared our method to the existing approach of Iwasaki and Shibata (2001), which entails an LMI conditions with quadratic parameter (δ and $\dot{\delta}$) dependence (Eq. (16) of Iwasaki and Shibata (2001)). Using a special relaxation technique based on the multiconvex property of the set of the feasible parameter values, the quadratically parameter dependent LMI can be can be ensured by a finite number of *parameter independent* LMIs. However, we should mention that this relaxation increases the solutions conservatism. For the sake of simplicity, instead of this kind of linearization, we tested the feasibility of the parameter dependent LMI on an appropriately dense grid of the parameter space $\mathcal{D} \times \check{\mathcal{D}}$. In Table 1, the number of uniformly distributed grid points is denoted by M.

4. DISCUSSION

In this contribution, we presented an alternative method for global stability analysis of LPV models by constructing parameter dependent rational Lyapunov functions. Differently from the known LPV solutions where the uncertainty

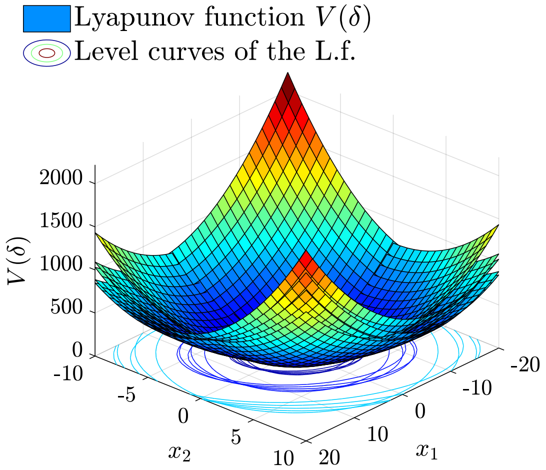

Fig. 1. Surface plot and level curves of the L.f. for five different values of the uncertain parameter $\delta_i \in \{-1, -0.5, 0, 0.5, 1\} \in \mathcal{D}$. The level curves point out the boundaries of the level sets $\varepsilon_\alpha(\delta_i) = \{x \mid V(x) \leq \alpha\}$, where $\alpha = 100, 300, 700$.

	Our method	Iwasaki and Shibata (2001)
LMIs	$2 \cdot (5 \times 5)$ $4 \cdot (11 \times 11)$	$1 \cdot (11 \times 11)$ $M \cdot (9 \times 9)$
Variables	162	186
Solver time	0.11 [sec]	if $M = 100$: 0.30 [sec] if $M = 49$: 0.14 [sec]

Table 1. Number and size of LMIs of our proposed method compared to algorithm of Iwasaki and Shibata (2001)

block Δ is separated from the state space representation (see eg. Seiler et al. (2009)), the LMI-based approach proposed by Trofino and Dezuo (2013) keeps the uncertain parameters inside the model. This method was further improved by Polcz et al. (2015, 2017, 2018) where the LMI conditions were generated automatically using symbolical operations, which resulted in a dimensionally reduced optimization problem compared to the original contribution of Trofino and Dezuo (2013). These computations were successfully demonstrated on higher (at most 5-) dimensional nonlinear quasi-LPV rational and polynomial systems in Polcz et al. (2016, 2017); Polcz and Szederkényi (2016). Therefore, the proposed method adapted for the global stability analysis of LPV system is promising from a computational point of view, but the proposed approach should be evaluated further mainly by comparing to other LPV solutions in the literature. The two main aspects during the comparison shall be the computational effort and the solution's conservatism.

REFERENCES

Cox, P.B., Weiland, S., and Tóth, R. (2018). Affine parameter-dependent Lyapunov functions for LPV systems with affine dependence. URL `arxiv.org/abs/1803.11543v2`.

Iwasaki, T. and Shibata, G. (2001). LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Transactions on Automatic Control*, 46(8), 1195–1208. doi:10.1109/9.940924.

Khargonekar, P.P., Petersen, I.R., and Zhou, K. (1990). Robust stabilization of uncertain linear systems: quadratic stabilizability and h infin; control theory. *IEEE Transactions on Automatic Control*, 35(3), 356–361. doi:10.1109/9.50357.

Löfberg, J. (2004). Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*. Taipei, Taiwan. doi:10.1109/CACSD.2004.1393890. URL `http://users.isy.liu.se/johanl/yalmip`.

Polcz, P., Szederkényi, G., and Hangos, K.M. (2016). Computational stability analysis of an uncertain bioreactor model. In *13th International Symposium on Stability, Vibration, and Control of Machines and Structures - SVCS 2016, June 16-18, Budapest, Hungary*, 21–32.

Polcz, P., Péni, T., and Szederkényi, G. (2017). Improved algorithm for computing the domain of attraction of rational nonlinear systems. *European Journal of Control*. doi:10.1016/j.ejcon.2017.10.003.

Polcz, P., Péni, T., and Szederkényi, G. (2018). Reduced linear fractional representation of nonlinear systems for stability analysis. In *9th Vienna International Conference on Mathematical Modelling*, accepted.

Polcz, P. and Szederkényi, G. (2016). Computational stability analysis of lotka-volterra systems. *Hungarian Journal of Industry and Chemistry*, 44(2), 113–120.

Polcz, P., Szederkényi, G., and Péni, T. (2015). An improved method for estimating the domain of attraction of nonlinear systems containing rational functions. In *Journal of Physics: Conference Series*, volume 659, 012038. IOP Publishing.

Scherer, C.W. and Weiland, S. (2000). *Linear Matrix Inequalities In Control*. Technical University of Delft, The Netherlands, Delft.

Seiler, P., Topcu, U., Packard, A., and Balas, G. (2009). Parameter-dependent Lyapunov functions for linear systems with constant uncertainties. *IEEE Transactions on Automatic Control*, 54(10), 2410–2416. doi:10.1109/TAC.2009.2029294. Cited By 7.

Sturm, J.F. (1999). Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4), 625–653. doi:10.1080/10556789908805766.

Trofino, A. and Dezuo, T.J.M. (2013). LMI stability conditions for uncertain rational nonlinear systems. *International Journal of Robust and Nonlinear Control*, 24(18), 3124–3169. doi:10.1002/rnc.3047. Cited By 14.

Vannelli, A. and Vidyasagar, M. (1985). Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica*, 21(1), 69–80. doi:10.1016/0005-1098(85)90099-8.

Wu, F. and Dong, K. (2006). Gain-scheduling control of lft systems using parameter-dependent Lyapunov functions. *Automatica*, 42(1), 39–50. doi:10.1016/j.automatica.2005.08.020. Cited By 127.

Wu, F. and Prajna, S. (2005). Sos-based solution approach to polynomial lpv system analysis and synthesis problems. *International Journal of Control*, 78(8), 600–611. doi:10.1080/00207170500114865.

Wu, F. (1995). *Control of Linear Parameter Varying Systems*. Ph.D. thesis, University of California at Berkeley.