Self-extensionality of finitely-valued logics

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SELF-EXTENSIONALITY OF FINITELY-VALUED LOGICS

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Abstract. We start from proving a general characterization of the self-extensionality of sentential logics implying the decidability of this problem as for (possibly, multiple) finitely-valued logics. And what is more, in case of finitely-valued logics with equality determinant as well as either implication or both conjunction and disjunction, we then derive a characterization yielding a quite effective algebraic criterion of checking their self-extensionality via analyzing homomorphisms between (viz., in the unitary case, endomorphisms of) their underlying algebras and equally being a quite useful heuristic tool, manual applications of which are demonstrated within the framework of Łukasiewicz' finitely-valued logics, four-valued expansions of Belnap's "useful" four-valued logic, their non-unitary three-valued extensions, unitary inferentially consistent non-classical ones being well-known to be non-self-extensional, as well as unitary three-valued disjunctive (in particular, implicative) logics with subclassical negation (including both paraconsistent and paracomplete ones).

1. Introduction

Recall that a sentential logic (cf., e.g., [8]) is said to be *self-extensional*, whenever its inter-derivability relation is a congruence of the formula algebra. Such feature is typical of both two-valued (in particular, classical) and super-intuitionistic logics as well as some interesting many-valued ones (like Belnap's "useful" four-valued one [3]). Here, we explore it laying a special emphasis onto the general framework of finitely-valued logics and the decidability issue with reducing the complexity of effective procedures of verifying it, when restricting our consideration by those logics of such a kind which possess certain peculiarities — both classical either implication or both conjunction and disjunction (in Tarski's conventional sense) and binary equality determinant in a sense extending [18] towards [19]. We then exemplify our universal elaboration by discussing four (perhaps, most representative) generic classes of logics of the kind involved: Łukasiewicz' finitely-valued logics [9], four-valued expansions of Belnap's logic (cf. [17]), their non-unitary three-valued extensions, unitary inferentially consistent non-classical ones being well-known (due to [19]) to be non-self-extensional, as well as unitary three-valued disjunctive (in particular, implicative) logics with subclassical negation (including both paraconsistent and paracomplete ones).

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set and Lattice Theory, Universal Algebra and Logic to be found, if necessary, in standard mathematical handbooks like [2, 5, 10]). Section 2 is a concise summary of particular basic issues underlying the paper, most of which, though having become a part of algebraic and logical folklore, are still recalled just for the exposition to be properly self-contained. In Section 3, we then develop/recall certain advanced generic issues concerning disjunctivity, implicativity and equality determinants. Section 4

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is a collection of main *general* results of the paper that are then exemplified in Section 5 (aside from Łukasiewicz' finitely-valued logics, whose non-self-extensionality has actually been due [19], as we briefly discuss within Example 4.16 — this equally concerns certain particular instances discussed in Section 5 and summarized in Example 4.17). Finally, Section 6 is a brief summary of principal contributions of the paper.

2. Basic issues

Notations like img, dom, ker, hom, π_i and Con and related notions are supposed to be clear.

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. Then, given any $(N \cup \{n\}) \subseteq \omega$, set $(N \div n) \triangleq \{\frac{m}{n} \mid m \in N\}$. The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations, while singletons are identified with their unique elements, unless any confusion is possible. A function f is said to be *singular*, provided $|\operatorname{img} f| \in 2$.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Then, an *enumeration of* S is any bijection from $|S|$ onto S. As usual, given any equivalence relation θ on S, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in the sequence $\bar{t}$ form, its s-th component (viz., the value under argument s), where $s \in S$, being written as t_s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s \in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a) \rangle_{s \in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a \rangle \mid a \in S\}$, functions of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i \in (\omega \setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples/sequences, the binary concatenation operation on which being denoted by $*$, as usual. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\operatorname{dom} \bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a} \upharpoonright (l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ\langle n \times \{f\}, \Delta_S \rangle) : S \to S$. Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi_S^T \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ *of* T *in* S.

Let A be a set. A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \varnothing$, when taking $S = \varnothing$. A subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* $\{$*transitive*$\}$ $\langle$*inductive/finitary/compact*$\rangle$, provided, for all $(B,)D \in \wp(A)$ $\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. A *closure operator over* A is any monotonic idempotent transitive operator over A, in which case $\operatorname{img} C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (including $\operatorname{img} C$

itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, called *dual to* C and vice versa. (Clearly, C is inductive iff $\operatorname{img} C$ is so.)

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives.*

Given a Σ-algebra $\mathfrak{A}$, $\mathrm{Con}(\mathfrak{A})$ is an inductive closure system over A^2 forming a bounded lattice with meet $\theta \cap \vartheta$ of any $\theta, \theta \in \mathrm{Con}(\mathfrak{A})$, their join $\theta \vee \vartheta$, being the transitive closure of $\theta \cup \vartheta$, zero Δ_A and unit A^2. Next, a *[partial] endomorphism of* $\mathfrak{A}$ is any homomorphism from [a subalgebra of] $\mathfrak{A}$ to $\mathfrak{A}$. Then, given a class K of Σ-algebras, set $\hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, in which case $\ker[\hom(\mathfrak{A}, \mathsf{K})] \subseteq \mathrm{Con}(\mathfrak{A})$, and so $(A^2 \cap \bigcap \ker[\hom(\mathfrak{A}, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{A})$.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$, put $\bar{x}_\alpha \triangleq \langle x_\beta \rangle_{\beta\in\alpha}$, $V_\alpha \triangleq (\operatorname{img} \bar{x}_\alpha)$, elements of which being viewed as *(propositional/sentential) variables of rank* α, and $(\forall_\alpha) \triangleq (\forall \bar{x}_\alpha)$. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ being called *(propositional/sentential)* Σ*-substitutions/-formulas of rank* α. A $\theta \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$ is said to be *fully invariant*, if, for every Σ-substitution σ of rank α, it holds that $\sigma[\theta] \subseteq \theta$. Recall that

$$\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [(\operatorname{img} h) = B) \Rightarrow] \\ (\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{B}) \supseteq [=]\{h \circ g \mid g \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\}), \quad (2.1)$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras. Any $\langle \phi, \psi \rangle \in \mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Fm}^\alpha_\Sigma)^2$ is referred to as a Σ*-equation/-indentity of rank* α and normally written in the standard equational form $\phi \approx \psi$. (In general, any mention of α is normally omitted, whenever $\alpha = \omega$.) In this way, given any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$, $\ker h$ is the set of all Σ-identities of rank α *true/satisfied in* $\mathfrak{A}$ *under* h. Likewise, given a class K of Σ-algebras, $\theta^\alpha_{\mathsf{K}} \triangleq (\mathrm{Eq}^\alpha_\Sigma \cap \bigcap \ker[\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$, being fully invariant, in view of (2.1), is the set of all all Σ-identities of rank α *true/satisfied in* K, in which case we set $\mathfrak{F}^\alpha_{\mathsf{K}} \triangleq (\mathfrak{Fm}^\alpha_\Sigma / \theta^\alpha_{\mathsf{K}})$. (In case both α as well as both K and all members of it are finite, the set $I \triangleq \{\langle h, \mathfrak{A} \rangle \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A}), \mathfrak{A} \in \mathsf{K}\}$ is finite — more precisely, $|I| = \sum_{\mathfrak{A}\in\mathsf{K}} |A|^\alpha$, in which case $g \triangleq (\prod_{i\in I} \pi_0(i)) \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \prod_{i\in I}(\pi_1(i) \upharpoonright \operatorname{img} \pi_0(i)))$ with $(\ker g) = \theta \triangleq \theta^\alpha_{\mathsf{K}}$, and so, by the Homomorphism Theorem, $e \triangleq (g \circ \nu_\theta^{-1})$ is an isomorphism from $\mathfrak{F}^\alpha_{\mathsf{K}}$ onto the subdirect product $(\prod_{i\in I}(\pi_1(i) \upharpoonright \operatorname{img} \pi_0(i))) \upharpoonright (\operatorname{img} g)$ of $\langle \pi_1(i) \upharpoonright \operatorname{img} \pi_0(i) \rangle_{i\in I}$. In this way, the former is finite, for the latter is so — more precisely, $|F^\alpha_{\mathsf{K}}| \leqslant (\max_{\mathfrak{A}\in\mathsf{K}} |A|)^{|I|}$.)

The class of all Σ-algebras satisfying every element of an $\mathcal{E} \subseteq \mathrm{Eq}^\omega_\Sigma$ is called the *variety axiomatized by* $\mathcal{E}$. Then, the variety $\mathbf{V}(\mathsf{K})$ axiomatized by $\theta^\omega_{\mathsf{K}}$ is the least variety including K and is said to be *generated by* K, in which case $\theta^\alpha_{\mathbf{V}(\mathsf{K})} = \theta^\alpha_{\mathsf{K}}$, and so $\mathfrak{F}^\alpha_{\mathbf{V}(\mathsf{K})} = \mathfrak{F}^\alpha_{\mathsf{K}}$.

Given a fully invariant $\theta \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$, by (2.1), $\mathfrak{Fm}^\omega_\Sigma/\theta$ belongs to the variety V axiomatized by θ, in which case any Σ-identity satisfied in V belongs to θ, and so $\theta^\omega_{\mathsf{V}} = \theta$. In particular, given a variety V of Σ-algebras, we have $\mathfrak{F}^\alpha_{\mathsf{V}} \in \mathsf{V}$. And what is more, given any $\mathfrak{A} \in \mathsf{V}$ and any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$, as $\theta \triangleq \theta^\alpha_{\mathsf{V}} \subseteq (\ker h)$, by the Homomorphism Theorem, $g \triangleq (h \circ \nu_\theta^{-1}) \in \hom(\mathfrak{F}^\alpha_{\mathsf{V}}, \mathfrak{A})$, in which case $h = (g \circ \nu_\theta)$, and so $\mathfrak{F}^\alpha_{\mathsf{V}}$ is a *free algebra of* V *with* $|\alpha|$ *free generators*, whenever V contains a non-one-element member, in which case $\nu_\theta \upharpoonright V_\alpha$ is injective, and so $|\alpha|$ is the cardinality of the set V_α/θ generating $\mathfrak{F}^\alpha_{\mathsf{V}}$, for V_α generates $\mathfrak{Fm}^\alpha_\Sigma$.

The mapping $\mathrm{Var} : \mathrm{Fm}^{\omega}_{\Sigma} \to \wp_{\omega}(V_{\omega})$ assigning the set of all *actually* occurring variables is defined in the standard recursive manner by induction on construction of Σ-formulas. Given any $m, n \in \omega$, the Σ-substitution extending $\Delta_{V_m} \cup [x_i/x_{i+n}]_{i\in(\omega\setminus m)}$ is denoted by $\sigma_{m:+n}$.

2.2.1. *Equational disjunctive systems.* According to [19, 21], a(n) *(equational) disjunctive system for* a class K of Σ-algebras is any $\mho \subseteq \mathrm{Eq}^4_{\Sigma}$ such that

$$(\exists j \in 2 : a_{2j} = a_{2j+1}) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i]_{i\in 4}), \tag{2.2}$$

for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$.

2.2.2. *Lattice-theoretic background.*
2.2.2.1. Semi-lattices. Let $\diamond$ be a (possibly, secondary) binary connective of Σ.

A Σ-algebra $\mathfrak{A}$ is called a $\diamond$*-semi-lattice*, provided it satisfies semilattice (viz., idempotencity, commutativity and associativity) identities for $\diamond$, in which case we have the partial ordering $\leq^{\mathfrak{A}}_{\diamond}$ on A, given by $(a \leq^{\mathfrak{A}}_{\diamond} b) \overset{\mathrm{def}}{\Longleftrightarrow} (a = (a \diamond^{\mathfrak{A}} b))$, for all $a, b \in A$. Then, in case the poset $\langle A, \leq^{\mathfrak{A}}_{\diamond}\rangle$ has the least element (viz., *zero*) [in particular, when A is finite], this is denoted by $\flat^{\mathfrak{A}}_{\diamond}$, while $\mathfrak{A}$ is referred to as a $\diamond$*-semi-lattice with zero (a)* (whenever $a = \flat^{\mathfrak{A}}_{\diamond}$).

Lemma 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\diamond$-semi-lattices with zero and $h \in \hom(\mathfrak{A}, \mathfrak{B})$. Suppose $h[A] = B$. Then, $h(\flat^{\mathfrak{A}}_{\diamond}) = \flat^{\mathfrak{B}}_{\diamond}$.*

Proof. Then, there is some $a \in A$ such that $h(a) = \flat^{\mathfrak{B}}_{\diamond}$, in which case $(a \diamond^{\mathfrak{A}} \flat^{\mathfrak{A}}_{\diamond}) = \flat^{\mathfrak{A}}_{\diamond}$, and so $h(\flat^{\mathfrak{A}}_{\diamond}) = (h(a) \diamond^{\mathfrak{B}} h(\flat^{\mathfrak{A}}_{\diamond})) = (\flat^{\mathfrak{B}}_{\diamond} \diamond^{\mathfrak{B}} h(\flat^{\mathfrak{A}}_{\diamond})) = \flat^{\mathfrak{B}}_{\diamond}$, as required. □

2.2.2.1.1. Implicative inner semilattices. Set $(x_0 \uplus_{\diamond} x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

A Σ-algebra $\mathfrak{A}$ is called an $\diamond$*-implicative inner semi-lattice*, provided it is a $\uplus_{\diamond}$-semilattice and satisfies the Σ-identities:

$$(x_0 \diamond x_0) \approx (x_1 \diamond x_1), \tag{2.3}$$

$$((x_0 \diamond x_0) \diamond x_1) \approx x_1, \tag{2.4}$$

in which case it is an $\uplus_{\diamond}$-semilattice with zero $a \diamond^{\mathfrak{A}} a$, for any $a \in A$.
2.2.2.2. Distributive lattices. Let $\barwedge$ and $\veebar$ be (possibly, secondary) binary connectives of Σ.

A Σ-algebra $\mathfrak{A}$ is called a *[distributive]* $(\barwedge, \veebar)$*-lattice*, provided it satisfies [distributive] lattice identities for $\barwedge$ and $\veebar$ (viz., semilattice identities for both $\barwedge$ and $\veebar$ as well as mutual [both] absorption [and distributivity] identities for them), in which case $\leq^{\mathfrak{A}}_{\barwedge}$ and $\leq^{\mathfrak{A}}_{\veebar}$ are inverse to one another, and so, in case $\mathfrak{A}$ is a $\veebar$-semilattice with zero (in particular, when A is finite), $\flat^{\mathfrak{A}}_{\veebar}$ is the greatest element (viz., *unit*) of the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge}\rangle$. Then, in case $\mathfrak{A}$ is a {distributive} $(\barwedge, \veebar)$-lattice, it is said to be that *with zero/unit (a)*, whenever it is a $(\barwedge/\veebar)$-semilattice with zero (a).

Let $\Sigma_{+[,01]} \triangleq \{\wedge, \vee[, \bot, \top]\}$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)]. Then, a $\Sigma_{+[,01]}$-algebra $\mathfrak{A}$ is called a *[bounded] (distributive) lattice*, whenever it is a (distributive) $(\wedge, \vee)$-lattice [with zero $\bot^{\mathfrak{A}}$ and unit $\top^{\mathfrak{A}}$] {cf., e.g., [2]}.

Given any $n \in (\omega \setminus 2)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain $n \div (n-1)$ ordered by $\leqslant$.
2.2.2.2.1. De Morgan lattices. Let $\Sigma_{+,\sim[,01]} \triangleq (\Sigma_{+[,01]} \cup \{\sim\})$ with unary $\sim$ (negation). Then, a *[bounded] De Morgan lattice* ([17]) is any $\Sigma_{+,\sim[,01]}$-algebra, whose $\Sigma_{+[,01]}$-reduct is a [bounded] distributive lattice and that satisfies the following $\Sigma_{+,\sim}$-identities:

$$\sim\sim x_0 \approx x_0, \tag{2.5}$$

$$\sim(x_0 \wedge x_1) \approx (\sim x_0 \vee \sim x_1), \tag{2.6}$$

$$\sim(x_0 \vee x_1) \approx (\sim x_0 \wedge \sim x_1), \tag{2.7}$$

By $\mathfrak{DM}_{4[,01]}$ we denote the [bounded] De Morgan lattice with $(\mathfrak{DM}_{4[,01]}\upharpoonright\Sigma_{+[,01]}) \triangleq \mathfrak{D}^2_{2[,01]}$ and $\sim^{\mathfrak{DM}_{4[,01]}}\langle i,j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i,j \in 2$.

2.3. **Propositional logics and matrices.** A *[finitary/unary] Σ-rule* is any couple $\langle\Gamma,\varphi\rangle$, where $\Gamma \in \wp_{[\omega/(2\backslash 1)]}(\mathrm{Fm}^\omega_\Sigma)$ and $\varphi \in \mathrm{Fm}^\omega_\Sigma$, normally written in the standard sequent form $\Gamma \vdash \varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. Any Σ-substitution σ determines the equally-denoted unary operation on $\wp_{[\omega/(2\backslash 1)]}(\mathrm{Fm}^\omega_\Sigma)$ given by $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$. As usual, Σ-rules without premises are called *Σ-axioms* and are identified with their conclusions. A[n] *[axiomatic] (finitary/unary) Σ-calculus* is then any set $\mathcal{C}$ of (finitary/unary) Σ-rules [without premises].

A *(propositional/sentential) Σ-logic* (cf., e.g., [8]) is any closure operator C over $\mathrm{Fm}^\omega_\Sigma$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$ and all $\sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$, that is, $\operatorname{img} C$ is closed under inverse Σ-substitutions, in which case we have the equivalence relation $\equiv^\alpha_C \triangleq \{\langle\phi,\psi\rangle \in \mathrm{Eq}^\alpha_\Sigma \mid C(\phi) = C(\psi)\}$, where $\alpha \in \wp_{\infty\backslash 1}(\omega)$, called the *inter-derivablity relation of C*, when $\alpha = \omega$. A *congruence of C* is any $\theta \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$ such that $\theta \subseteq \equiv^\omega_C$, the set of all them being denoted by $\mathrm{Con}(C)$. Then, given any $\theta, \vartheta \in \mathrm{Con}(C)$, the transitive closure $\theta \vee \vartheta$ of $\theta \cup \vartheta$, being a congruence of $\mathfrak{Fm}^\omega_\Sigma$, is then that of C, for θ^ω_C, being an equivalence relation, is transitive. In particular, any maximal congruence of C (that exists, by Zorn Lemma, because $\mathrm{Con}(C) \ni \Delta_{\mathrm{Fm}^\omega_\Sigma}$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$ is so) is the greatest one to be denoted by $\eth(C)$, the variety $\mathrm{IV}(C)$ axiomatized by it being called the *intrinsic variety of C* (cf. [16]). Then, C is said to be *self-extensional*, whenever $\equiv^\omega_C \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$, in which case $\eth(C) = \equiv^\omega_C$. Next, C is said to be *[inferentially] (in)consistent*, if $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$ [(in which case $\equiv^\omega_C = \mathrm{Eq}^\omega_\Sigma \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$, and so C is self-extensional)]. Further, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied in/by C*, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of C*. Next, a Σ-logic C' is said to be a *(proper) [K-]extension of C* [where $K \subseteq \infty$], whenever $(C[\upharpoonright\wp_K(\mathrm{Fm}^\omega_\Sigma)]) \subseteq (\subsetneq)(C'[\upharpoonright\wp_K(\mathrm{Fm}^\omega_\Sigma)])$, in which case C is said to be a *(proper) [K-]sublogic of C'*. In that case, a[n axiomatic] Σ-calculus $\mathcal{C}$ is said to *axiomatize C' (relatively to C)*, if C' is the least Σ-logic (being an extension of C and) satisfying every rule in $\mathcal{C}$ [(in which case it is called an *axiomatic extension of C*)]. Furthermore, we have the finitary sublogic $C_\exists$ of C, defined by $C_\exists(X) \triangleq (\bigcup C[\wp_\omega(X)])$, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$, called the *finitarization of C*. Then, the extension of any finitary (in particular, diagonal) Σ-logic relatively axiomatized by a finitary Σ-calculus is a sublogic of its own finitarization, in which case it is equal to this, and so is finitary (in particular, the Σ-logic axiomatized by a finitary Σ-calculus is finitary; conversely, any [finitary] Σ-logic is axiomatized by the [finitary] Σ-calculus consisting of all those [finitary] Σ-rules, which are satisfied in C). Further, C is said to be *$\bar\wedge$-conjunctive*, where $\bar\wedge$ is a (possibly, secondary) binary connective of Σ (tacitly fixed throughout the paper), provided $C(\phi\bar\wedge\psi) = C(\{\phi,\psi\})$, for all $\phi,\psi \in \mathrm{Fm}^\omega_\Sigma$, in which case any extension of C is so. Likewise, C is said to be *$\veebar$-disjunctive*, where $\veebar$ is a (possibly, secondary) binary connective of Σ (tacitly fixed throughout the paper), provided $C(X\cup\{\phi\veebar\psi\}) = (C(X\cup\{\phi\})\cap C(X\cup\{\psi\}))$, where $(X \cup \{\phi,\psi\}) \subseteq \mathrm{Fm}^\omega_\Sigma$, in which case the following rules:

$$x_i \vdash (x_0 \veebar x_1), \tag{2.8}$$

$$(x_0 \veebar x_1) \vdash (x_1 \veebar x_0), \tag{2.9}$$

$$(x_0 \veebar x_0) \vdash x_0, \tag{2.10}$$

where $i \in 2$, are satisfied in C, and so in its extensions. Furthermore, C is said to *have Deduction Theorem (DT) with respect to* a (possibly, secondary) binary connective $\sqsupset$ of Σ (tacitly fixed throughout the paper), provided, for all $\phi \in X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(X)$, it holds that $(\phi \sqsupset \psi) \in C(X \setminus \{\phi\})$, Then, C is said to be *weakly $\sqsupset$-implicative*, if it has DT with respect to $\sqsupset$ and satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1, \tag{2.11}$$

in which case the following axioms:

$$x_0 \sqsupset x_0, \tag{2.12}$$

$$x_0 \sqsupset (x_1 \sqsupset x_0), \tag{2.13}$$

$$(x_0 \sqsupset (x_1 \sqsupset x_2)) \sqsupset ((x_0 \sqsupset x_1) \sqsupset (x_0 \sqsupset x_2)) \tag{2.14}$$

are satisfied in C. Likewise, C is said to be *(strongly) $\sqsupset$-implicative*, whenever it is weakly so as well as satisfies the *Peirce Law* axiom (cf. [11]):

$$(((x_0 \sqsupset x_1) \sqsupset x_0) \sqsupset x_0). \tag{2.15}$$

Next, C is said to *have Property of Weak Contraposition (PWC) with respect to* a unary $\sim \in \Sigma$ (tacitly fixed throughout the paper), provided, for all $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(\phi)$, it holds that $\sim\phi \in C(\sim\psi)$. Then, C is said to be *$\sim$-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1. \tag{2.16}$$

Likewise, C is said to be *$(\veebar, \sim)$-paracomplete*, whenever it does not satisfy the *Excluded Middle Law* axiom:

$$x_0 \veebar \sim x_0. \tag{2.17}$$

Finally, C is said to be *theorem-less/purely-inferential*, whenever it has no theorem, that is, $\varnothing \in (\mathrm{img}\, C)$. In general, $(\mathrm{img}\, C) \cup \{\varnothing\}$ is closed under inverse Σ-substitutions, for $\mathrm{img}\, C$ is so, in which case the dual closure operator C_{+0} is the greatest purely-inferential sublogic of C, called the *purely-inferential/theorem-less version of* C, while

$$\equiv^{\omega}_{C} = \equiv^{\omega}_{C_{+0}}, \tag{2.18}$$

and so C_{+0} is self-extensional iff C is so.

A *(logical) Σ-matrix* (cf. [8]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of $A[\cap D^{\mathcal{A}}]$ being referred to as *[distinguished] values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding Fraktur letters [with same indices, if any].) This is said to be *n-valued/[in]consistent/truth(-non)-empty/truth-|false-{non-}singular*, where $n \in (\omega \setminus 1)$, provided $(|A| = n)/(D^{\mathcal{A}} \neq [=]A)/(D^{\mathcal{A}} = (\neq)\varnothing)/(|(D^{\mathcal{A}}|(A \setminus D^{\mathcal{A}}))| \in \{\notin\}2)$, respectively. Next, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* its *Σ'-reduct* $(\mathcal{A} \upharpoonright \Sigma') \triangleq \langle \mathfrak{A} \upharpoonright \Sigma', D^{\mathcal{A}} \rangle$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, $\mathcal{A}$ is said to be *finite[ly generated]/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ dual to the closure system with basis $\{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.19}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Then, by (2.1), $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic, called the *logic of/defined by* M. A Σ-logic is said to be *(unitary) n-valued*, where $n \in (\omega \setminus 1)$, whenever it is defined by an n-valued Σ-matrix, in which case it is finitary (cf. [8]), and so is the logic of any finite class of finite Σ-matrices.

Remark 2.2. Given any class of Σ-matrices M and any truth-empty Σ-matrix $\mathcal{A}$, $\mathsf{M} \cup \{\mathcal{A}\}$ defines the theorem-less version of the logic of M. □

As usual, Σ-matrices are treated as first-order model structures of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any Σ-rule $\Gamma \vdash \phi$ being viewed as (the universal closure of; depending upon the context) the infinitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective]* $\{$*matrix*$\}$ *homomorphism from* $\mathcal{A}$ *[on]to* $\mathcal{B}$ is any $h \in \mathrm{hom}(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=)h^{-1}[D^{\mathcal{B}}]$ ([in which case $\mathcal{B}/\mathcal{A}$ is said to be a *strict surjective* $\{$*matrix*$\}$ *homomorphic image/counter-image of* $\mathcal{A}/\mathcal{B}$, respectively]), the set of all them being denoted by $\mathrm{hom}^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Then, by (2.1), we have:

$$(\exists h \in \mathrm{hom}^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=]\, \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.20}$$

for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$. Further, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B} {\restriction} A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$.

Given a Σ-matrix $\mathcal{A}$, $\chi^{\mathcal{A}} \triangleq \chi^{D^{\mathcal{A}}}_{A}$ is referred to as the *characteristic function of* $\mathcal{A}$. Then, any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq \theta^{\mathcal{A}} \triangleq (\ker \chi^{\mathcal{A}})$, in which case ν_θ is a strict surjective homomorphism from $\mathcal{A}$ onto $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, is called a *congruence of* $\mathcal{A}$, the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$. Given any $\theta, \vartheta \in \mathrm{Con}(\mathcal{A})$, the transitive closure $\theta \vee \vartheta$ of $\theta \cup \vartheta$, being a congruence of $\mathfrak{A}$, is then that of $\mathcal{A}$, for $\theta^{\mathcal{A}}$, being an equivalence relation, is transitive. In particular, any maximal congruence of $\mathcal{A}$ (that exists, by Zorn Lemma, because $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{A})$ is so) is the greatest one to be denoted by $\partial(\mathcal{A})$. Finally, $\mathcal{A}$ is said to be *[hereditarily] simple*, provided it has no non-diagonal congruence [and no non-simple submatrix].

Remark 2.3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* Σ*-matrices and* $h \in \mathrm{hom}^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$*. Then,* $\theta^{\mathcal{A}} = h^{-1}[\theta^{\mathcal{B}}]$ [and $h[\theta^{\mathcal{A}}] = \theta^{\mathcal{B}}$], while, for all $\theta \in \mathrm{Con}(\mathfrak{B})$, $h^{-1}[\theta] \in \mathrm{Con}(\mathfrak{A})$ [and $\theta = h[h^{-1}[\theta]]$, whereas, for all $\vartheta \in \mathrm{Con}(\mathfrak{A})$ including $\ker h$, $h[\vartheta] \in \mathrm{Con}(\mathfrak{B})$ and $\vartheta = h^{-1}[h[\vartheta]]$]. Therefore,

(i) $h^{-1}[\theta] \in \mathrm{Con}(\mathcal{A})$*, for all* $\theta \in \mathrm{Con}(\mathcal{B})$ *[while* $h[\vartheta] \in \mathrm{Con}(\mathcal{B})$*, for all* $\vartheta \in \mathrm{Con}(\mathcal{A})$ *including* $\ker h$*].*

In particular (when $\theta = \Delta_B$), $(\ker h) = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, in which case $(\ker h) \subseteq \partial(\mathcal{A})$, and so

(ii) h *is injective, whenever* $\mathcal{A}$ *is simple.*

[Moreover, when $\vartheta = \partial(\mathcal{A})$ and $\theta = \partial(\mathcal{B})$, we have $h^{-1}[\theta] \subseteq \vartheta \supseteq (\ker h)$, in which case we get $\theta = h[h^{-1}[\theta]] \subseteq h[\vartheta] \subseteq \theta$, and so $\theta = h[\vartheta]$, in which case $\vartheta = h^{-1}[h[\vartheta]] = h^{-1}[\theta]$, and so

(iii) $\partial(\mathcal{B}) = h[\partial(\mathcal{A})]$ *and* $\partial(\mathcal{A}) = h^{-1}[\partial(\mathcal{B})]$.

In particular (when $\mathcal{B} = (\mathcal{A}/\partial(\mathcal{A}))$ and $h = \nu_{\partial(\mathcal{A})}$), we have $h[\partial(\mathcal{A})] = h[\ker h] = \Delta_B$, and so

(iv) $\mathcal{A}/\partial(\mathcal{A})$ *is simple.*] □

A Σ-matrix $\mathcal{A}$ is said to be a *[K-]model of* a (finitary) Σ-logic C {*over* $\mathfrak{A}$} [where $K \subseteq \infty$], provided C is a $[K$-]sublogic of the logic of $\mathcal{A}$, the class of all ⟨simple of⟩ them being denoted by $\mathrm{Mod}^{\langle * \rangle}_{[K]}(C\{,\mathfrak{A}\})$, respectively. Then, $\pi_1[\mathrm{Mod}(C,\mathfrak{A})]$ is a(n inductive) closure system over A, the dual (finitary) closure operator being denoted by $\mathrm{Fg}^{\mathfrak{A}}_C$, in which case $\pi_1[\mathrm{Mod}(C,\mathfrak{Fm}^\omega_\Sigma)] = (\mathrm{img}\, C)$, and so $\mathrm{Fg}^{\mathfrak{Fm}^\omega_\Sigma}_C = C$ (while, given any finitary axiomatization $\mathcal{C}$ of C and any $(X \cup \{a\}) \subseteq A$, it holds that $a \in \mathrm{Fg}^{\mathfrak{A}}_C(X)$ iff a is *derivable in* $\mathcal{C}$ *from* X *over* $\mathfrak{A}$ in the sense that there is a(n) *(abstract)* $\mathcal{C}$-*derivation of* a *from* X *over* $\mathfrak{A}$, that is, any $\bar{b} \in A^+$ such that $a \in (\mathrm{img}\, \bar{b})$ and, for each $i \in (\mathrm{dom}\, \bar{b})$, either $b_i \in X$ or there are some $(\Gamma \vdash \varphi) \in \mathcal{C}$ and some $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $b_i = h(\varphi)$ and $h[\Gamma] \subseteq (\mathrm{img}(\bar{b}\restriction i))$ — the reservation "from X"/"over $\mathfrak{A}$" is omitted, whenever $(X = \varnothing)/(\mathfrak{A} = \mathfrak{Fm}^\omega_\Sigma)$, respectively; cf. [13]). Next, $\mathcal{A}$ is said to be $\sim$*-paraconsistent/*$(\veebar, \sim)$*-paracomplete*, whenever the logic of $\mathcal{A}$ is so. Further, $\mathcal{A}$ is said to be $\diamond$*-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a,b\} \subseteq D^{\mathcal{A}}) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, the logic of $\mathcal{A}$ is $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be $\diamond$*-disjunctive*, whenever $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}} \rangle$ is $\diamond$-conjunctive, in which case the logic of $\mathcal{A}$ is $\diamond$-disjunctive, and so is the logic of any class of $\diamond$-disjunctive Σ-matrices. Likewise, $\mathcal{A}$ is said to be $\diamond$*-implicative*, whenever $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, in which case it is $\uplus_\diamond$-disjunctive, while the logic of $\mathcal{A}$ is $\diamond$-implicative, for both (2.11) and (2.15) $= ((x_0 \sqsupset x_1) \uplus_\sqsupset x_0)$ are true in any $\sqsupset$-implicative (and so $\uplus_\sqsupset$-disjunctive) Σ-matrix, while DT is immediate, and so is the logic of any class of $\diamond$-implicative Σ-matrices. Finally, given any (possibly secondary) unary connective $\wr$ of Σ, put $(x_0 \diamond^\wr x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$. Then, $\mathcal{A}$ is said to be *(classically)* $\wr$*-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is consistent, and so truth-non-empty.

Remark 2.4. Let $\diamond$ and $\wr$ be as above. Then, the following hold:

(i) any $\wr$-negative Σ-matrix:
 a) is $\diamond$-disjunctive/-conjunctive iff it is $\diamond^\wr$-conjunctive/-disjunctive, respectively;
 b) defines a logic having PWC with respect to $\wr \in \Sigma$;

(ii) given any two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ and any $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{B})$, $\mathcal{A}$ is $\wr$-negative|$\diamond$-conjunctive/-disjunctive/-implicative if[f] $\mathcal{B}$ is so. □

Given a class M of Σ-matrices, the class of all strict surjective homomorphic [counter-]images/(consistent) submatrices of members of M is denoted, respectively, by $(\mathbf{H}^{[-1]}/\mathbf{S}_{(*)})(\mathsf{M})$.

Lemma 2.5. *Let* M *be a class of* Σ*-matrices. Then,* $\mathbf{H}(\mathbf{H}^{-1}(\mathsf{M})) \subseteq \mathbf{H}^{-1}(\mathbf{H}(\mathsf{M}))$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, $\mathcal{C} \in \mathsf{M}$ and $(h|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{C}|\mathcal{A})$. Then, by Remark 2.3(i), $(\ker(h|g)) \in \mathrm{Con}(\mathcal{B})$, in which case $(\ker(h|g)) \subseteq \theta \triangleq \partial(\mathcal{B}) \in \mathrm{Con}(\mathcal{B})$, and so, by the Homomorphism Theorem, $(\nu_\theta \circ (h|g)^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{C}|\mathcal{A}, \mathcal{B}/\theta)$. □

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^\alpha_{\Sigma'} \subseteq \mathrm{Fm}^\alpha_\Sigma$ and $\hom(\mathfrak{Fm}^\alpha_{\Sigma'}, \mathfrak{Fm}^\alpha_{\Sigma'}) = \{h \restriction \mathrm{Fm}^\alpha_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{Fm}^\alpha_\Sigma), h[\mathrm{Fm}^\alpha_{\Sigma'}] \subseteq \mathrm{Fm}^\alpha_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^\omega_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^\omega_{\Sigma'}$, called the Σ'*-fragment of* C, in which case C is said to be a *(*Σ*-)expansion of* C'. In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M} \restriction \Sigma'$.

2.3.1. *Classical matrices and logics.* A two-valued Σ-matrix $\mathcal{A}$ is said to be $\sim$*-classical*, whenever it is $\sim$-negative, in which case it is both consistent and truth-non-empty, and so is both false- and truth-singular, the unique element of $(A \setminus$

$D^{\mathcal{A}})/D^{\mathcal{A}}$ being denoted by $(0/1)_{\mathcal{A}}$, respectively (the index ${}_{\mathcal{A}}$ is often omitted, unless any confusion is possible), in which case $A = \{0,1\}$, while $\sim^{\mathfrak{A}} i = (1-i)$, for each $i \in 2$, whereas $\theta^{\mathcal{A}}$ is diagonal, for $\chi^{\mathcal{A}}$ is so, and so $\mathcal{A}$ is simple (in particular, hereditarily so, for it has no proper submatrix) but is not $\sim$-paraconsistent.

A Σ-logic is said to be *$\sim$-[sub]classical*, whenever it is [a sublogic of] the logic of a $\sim$-classical Σ-matrix. Then, $\sim$ is called a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{2.21}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd.

3. Preliminary key adnanced generic issues

3.1. **Congruence and equality determinants versus matrix simplicity and intrinsic varieties.** A *[binary] relational Σ-scheme* is any Σ-calculus [of the form] $\varepsilon \subseteq (\wp(\mathrm{Fm}_{\Sigma}^{[2\cap]\omega}) \times \mathrm{Fm}_{\Sigma}^{[2\cap]\omega})$, in which case, given any Σ-matrix $\mathcal{A}$, we set $\theta_{\varepsilon}^{\mathcal{A}} \triangleq \{\langle a,b\rangle \in A^2 \mid \mathcal{A} \models (\forall_{([2\cap]\omega)\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]\} \subseteq A^2$. Given a one more Σ-matrix $\mathcal{B}$ and an $h \in \hom_{\mathrm{S}}^{(\mathrm{S})}(\mathcal{A},\mathcal{B})$, we have:

$$h^{-1}[\theta_{\varepsilon}^{\mathcal{B}}] \subseteq (=)[=]\theta_{\varepsilon}^{\mathcal{A}}. \tag{3.1}$$

A *[unary] unitary relational Σ-scheme* is any $\Upsilon \subseteq \mathrm{Fm}_{\Sigma}^{[1\cap]\omega}$, in which case we have the unary [binary] relational Σ-scheme $\varepsilon_{\Upsilon} \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \sigma_{1:+1}[\Upsilon]\}$ such that $\theta_{\varepsilon_{\Upsilon}}^{\mathcal{A}}$, where $\mathcal{A}$ is any Σ-matrix, is an equivalence relation on A.

A *[binary] congruence/equality determinant for* a class of Σ-matrices M is any [binary] relational Σ-scheme ε such that, for each $\mathcal{A} \in \mathsf{M}$, $\theta_{\varepsilon}^{\mathcal{A}} \in \mathrm{Con}(\mathcal{A})/ = \Delta_A$, respectively, that includes a finite one, whenever both M and all members of it are finite.

Then, according to [19]/[18], a *[unary] unitary congruence/equality determinant for* a class of Σ-matrices M is any [unary] unitary relational Σ-scheme Υ such that ε_{Υ} is a/an congruence/equality determinant for M that includes a finite one, whenever both M and all members of it are finite. (It is unary unitary equality determinants that are equality determinants in the sense of [18].)

Lemma 3.1 (cf., e.g., [19])**.** $\mathrm{Fm}_{\Sigma}^{\omega}$ *is a unitary congruence determinant for every Σ-matrix $\mathcal{A}$.*

Proof. We start from proving the fact the equivalence relation $\theta \triangleq \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}_{\Sigma}^{\omega}}} \in \mathrm{Con}(\mathfrak{A})$. For consider any $\varsigma \in \Sigma$ of arity $n \in \omega$, any $i \in n$, in which case $n \neq 0$, any $\vec{a} \in \theta$, any $\bar{b} \in A^{n-1}$, any $\phi \in \mathrm{Fm}_{\Sigma}^{\omega}$ and any $\bar{c} \in A^{\omega}$. Put $\psi \triangleq \varsigma(\langle\langle x_{j+1}\rangle_{j\in i}, x_0\rangle * \langle x_{k+1}\rangle_{k\in(n\setminus i)})$ and $\varphi \triangleq ((\sigma_{1:+n}\phi)[x_0/\psi]) \in \mathrm{Fm}_{\Sigma}^{\omega}$. Then, we have

$$\begin{aligned}
&(\sigma_{1:+1}\phi)^{\mathfrak{A}}[x_{l+1}/c_l; x_0/\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_0\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})]_{l\in\omega} = \\
&\qquad (\sigma_{1:+1}\varphi)^{\mathfrak{A}}[x_{l+n+1}/c_l; x_0/a_0; x_{m+1}/b_m]_{l\in\omega; m\in(n-1)} \in D^{\mathcal{A}} \Leftrightarrow \\
&\qquad D^{\mathcal{A}} \ni (\sigma_{1:+1}\varphi)^{\mathfrak{A}}[x_{l+n+1}/c_l; x_0/a_1; x_{m+1}/b_m]_{l\in\omega; m\in(n-1)} = \\
&\qquad\qquad (\sigma_{1:+1}\phi)^{\mathfrak{A}}[x_{l+1}/c_l; x_0/\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_1\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})]_{l\in\omega},
\end{aligned}$$

in which case we eventually get

$$\langle \varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_0\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)}), \varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_1\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})\rangle \in \theta,$$

and so $\theta \in \mathrm{Con}(\mathfrak{A})$. Finally, as $x_0 \in \mathrm{Fm}_{\Sigma}^{\omega}$, we clearly have $\theta \subseteq \theta^{\mathcal{A}}$, as required. □

Corollary 3.2. *Let C be a Σ-logic, $\theta \in \mathrm{Con}(C)$, $\mathcal{A} \in \mathrm{Mod}(C)$ and $h \in \hom(\mathfrak{Fm}_{\Sigma}^{\omega}, \mathfrak{A})$. Then, $h[\theta] \subseteq \partial(\mathcal{A})$.*

Proof. Consider any $\langle \phi, \psi \rangle \in \theta$, any $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $g(x_{0/1}) = h(\phi/\psi)$ and any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, $V \triangleq (\mathrm{Var}(\sigma_{1:+1}(\varphi)) \setminus \{x_0\}) \in \wp_{\omega}(V_{\omega})$. Let $n \triangleq |V| \in \omega$ and $\bar{v}$ any enumeration of V. Likewise, $U \triangleq (\bigcup \mathrm{Var}[\{\phi, \psi\}]) \in \wp_{\omega}(V_{\omega})$, in which case $V_{\omega} \setminus U$ is infinite, and so there is an injective $\bar{u} \in (V_{\omega} \setminus U)^n$. Then, by the reflexivity of $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, we have $\xi \triangleq (\sigma_{1:+1}(\varphi)[x_0/\phi; v_i/u_i]_{i \in n})\ \theta\ \eta \triangleq (\sigma_{1:+1}(\varphi)[x_0/\psi; v_i/u_i]_{i \in n})$. Let $f \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h \upharpoonright (V_{\omega} \setminus (\mathrm{img}\, \bar{u}))) \cup [u_i/g(v_i)]_{i \in n}$. Then, as $\mathcal{A} \in \mathrm{Mod}(C)$ and $\theta \subseteq \equiv^{\omega}_{C}$, we get $g(\sigma_{1:+1}(\varphi)) = f(\xi)\ \theta^{\mathcal{A}}\ f(\eta) = g(\sigma_{1:+1}(\varphi)[x_0/x_1])$. In this way, $h(\phi)\ \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}\ h(\psi)$, and so Lemma 3.1 completes the argument. □

As a particular case of Corollary 3.2, we first have:

Corollary 3.3. *Let C be a Σ-logic. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathrm{IV}(C)$.*

Corollary 3.4. *Let C be a Σ-logic. Then, $\eth(C)$ is fully invariant. In particular, $\eth(C) = \theta^{\omega}_{\mathrm{IV}(C)}$.*

Proof. Consider any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ and any $T \in (\mathrm{img}\, C)$, in which case, by the structurality of C, $\mathcal{A}_T \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T \rangle \in \mathrm{Mod}(C)$, and so, by Corollary 3.2, $\sigma[\eth(C)] \subseteq \eth(\mathcal{A}_T)$. Thus, $\sigma[\eth(C)] \subseteq \theta \triangleq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap \{\eth(\mathcal{A}_T) \mid T \in (\mathrm{img}\, C)\}) \subseteq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap \{\theta^{\mathcal{A}_T} \mid T \in (\mathrm{img}\, C)\}) = \equiv^{\omega}_{C}$. Moreover, for each $T \in (\mathrm{img}\, C)$, $\eth(\mathcal{A}_T) \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, in which case $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so $\sigma[\eth(C)] \subseteq \theta \subseteq \eth(C)$. □

Lemma 3.5. *Let M be a class of Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\theta^{\omega}_{\mathsf{K}} \subseteq \equiv^{\omega}_{C}$, in which case $\theta^{\omega}_{\mathsf{K}} \subseteq \eth(C)$, and so $\mathrm{IV}(C) \subseteq \mathbf{V}(\mathsf{K})$.*

Proof. Then, for any $\langle \phi, \psi \rangle \in \theta^{\omega}_{\mathsf{K}}$, each $\mathcal{A} \in \mathsf{M}$ and all $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, $\mathfrak{A} \in \mathsf{K}$, in which case $\langle h(\phi), h(\psi) \rangle \in \Delta_A \subseteq \theta^{\mathcal{A}}$, and so $\phi \equiv^{\omega}_{C} \psi$, as required. □

By Corollary 3.3 and Lemma 3.5, we immediately have:

Corollary 3.6. *Let M be a class of Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathbf{V}(\mathsf{K})$.*

Likewise, by Corollary 3.3 and Lemma 3.5, we also have:

Theorem 3.7. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\mathrm{IV}(C) = \mathbf{V}(\mathsf{K})$.*

Lemma 3.8. *Let $\mathcal{A}$ be a Σ-matrix and ε a congruence determinant for $\mathcal{A}$. Then, $\eth(\mathcal{A}) = \theta^{\mathcal{A}}_{\varepsilon}$. In particular, $\mathcal{A}$ is simple iff ε is an equality determinant for it.*

Proof. Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b \rangle \in \theta$. Then, as $\mathrm{Con}(\mathcal{A}) \ni \theta^{\mathcal{A}}_{\varepsilon} \supseteq \Delta_A \ni \langle a, a \rangle$, we have $\mathcal{A} \models (\forall_{\omega \setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/a]$, in which case, by the reflexivity of θ, we get $\mathcal{A} \models (\forall_{\omega \setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]$, and so $\langle a, b \rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, as required. □

Lemma 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, ε a/an congruence/equality determinant for $\mathcal{B}$ and h a/an strict homomorphism/embedding from/of $\mathcal{A}$ to/into $\mathcal{B}$. Suppose either ε is binary or $h[A] = B$. Then, ε is a/an congruence/equality determinant for $\mathcal{A}$.*

Proof. In that case, by (3.1), we have $\theta^{\mathcal{A}}_{\varepsilon} = h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}]$. In this way, Remark 2.3(i)/"the injectivity of h" completes the argument. □

Theorem 3.10. *Let $\mathcal{A}$ be a Σ-matrix. Then, the following are equivalent:*

(i) *$\mathcal{A}$ is hereditarily simple;*
(ii) *$\mathcal{A}$ has a binary equality determinant;*
(iii) *$\mathcal{A}$ has a unary binary equality determinant.*

Proof. First, (ii) is a particular case of (iii), (ii)⇒(i) being by Lemmas 3.8 and 3.9.

Finally, assume (i) holds. Consider any $a, b \in A$. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}$ generated by $\{a, b\}$. Then, it is simple, by (i). Therefore, by Lemmas 3.1 and 3.8, $\Delta_B = \theta^{\mathcal{B}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}$. On the other hand, we have the unary binary relational Σ-scheme $\varepsilon \triangleq (\bigcup\{\sigma[\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}] \mid \sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{2}_{\Sigma}), (\sigma{\restriction}V_2) = \Delta_{V_2}\})$ such that $(\langle a, b\rangle \in \theta^{\mathcal{B}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}) \Leftrightarrow (\langle a, b\rangle \in \theta^{\mathcal{B}}_{\varepsilon})$, for $\mathfrak{B}$ is generated by $\{a, b\}$. In this way, by (3.1) with $h = \Delta_B$ as well as $\mathcal{A}$ and $\mathcal{B}$ instead of one another, we get $(a = b) \Leftrightarrow (\langle a, b\rangle \in \theta^{\mathcal{B}}_{\varepsilon}) \Leftrightarrow (\langle a, b\rangle \in \theta^{\mathcal{A}}_{\varepsilon})$. Thus, ε is an equality determinant for $\mathcal{A}$, and so (iii) holds, as required. □

Lemma 3.11. *Let $\mathcal{A}$ be a Σ-matrix with unary unitary equality determinant Υ, $\mathcal{B}$ a submatrix of $\mathcal{A}$ and $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. Then, h is diagonal.*

Proof. Consider any $a \in B$. Then, for any $v \in \Upsilon$, we have $(v^{\mathfrak{A}}(a) = v^{\mathfrak{B}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (v^{\mathfrak{A}}(h(a)) = h(v^{\mathfrak{B}}(a)) \in D^{\mathcal{A}})$, so we get $h(a) = a$, as required. □

3.2. **Disjunctivity.** Fix any set A, any closure operator C over A and any $\delta : A^2 \to A$, in which case we set $\delta(X, Y) \triangleq \delta[X \times Y]$, for all $X, Y \subseteq A$.

Then, C is said to be *δ-disjunctive*, provided, for all $a, b \in A$ and every $X \subseteq A$, it holds that

$$C(X \cup \{\delta(a, b)\}) = (C(X \cup \{a\}) \cap C(X \cup \{b\})), \tag{3.2}$$

in which case the following clearly hold, by (3.2) with $X = \varnothing$:

$$\delta(a, b) \in C(a), \tag{3.3}$$
$$\delta(a, b) \in C(b), \tag{3.4}$$
$$a \in C(\delta(a, a)), \tag{3.5}$$
$$\delta(b, a) \in C(\delta(a, b)), \tag{3.6}$$

and so, by (3.3), (3.4) and (3.2), does:

$$\delta(C(X \cup \{b\}), a) \subseteq C(X \cup \{\delta(b, a)\}). \tag{3.7}$$

Conversely, we have:

Lemma 3.12. *Suppose either* (3.3) *or* (3.4) *as well as both* (3.5), (3.6) *and* (3.7) *hold. Then, C is δ-disjunctive.*

Proof. In that case, by (3.6), both (3.3) and (3.4) hold, and so does the inclusion from left to right in (3.2). Conversely, consider any $c \in (C(X \cup \{b\}) \cap C(X \cup \{a\}))$, where $(X \cup \{a, b\}) \subseteq A$. Then, by (3.6) and (3.7), we have $\delta(b, c) \in C(X \cup \{\delta(a, b)\})$. Likewise, by (3.5) and (3.7), we have $c \in C(X \cup \{\delta(b, c)\})$. Therefore, we eventually get $c \in C(X \cup \{\delta(a, b)\})$, as required. □

3.2.1. *Disjunctive models of finitely-valued disjunctive logics.*

Lemma 3.13. *Let M be a finite class of finite Σ-matrices and $\mathcal{A}$ a finitely-generated (in particular, finite) consistent $\veebar$-disjunctive model of the logic of M. Suppose* (2.8) *with $i \in 2$ are true in M (in particular, all members of it are $\veebar$-disjunctive). Then, $\mathcal{A} \in \mathbf{H}(\mathbf{H}^{-1}(\mathbf{S}_*(\mathsf{M})))$.*

Proof. Take any $A' \in \wp_{\omega\setminus 1}(A)$ generating $\mathfrak{A}$. In that case, $n \triangleq |A'| \in (\omega \setminus 1) \subseteq \wp_{\infty\setminus 1}(\omega)$. Let $h \in \hom(\mathfrak{Fm}^{n}_{\Sigma}, \mathfrak{A})$ extend any bijection from V_n onto A', in which case $(\operatorname{img} h) = A$, and so h is a strict surjective homomorphism from $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{n}_{\Sigma}, T\rangle$ onto $\mathcal{A}$, where $T \triangleq h^{-1}[D^{\mathcal{A}}]$. Then, as $\mathcal{A}$ is consistent, by (2.19), we have $\mathrm{Fm}^{n}_{\Sigma} \supsetneq T \supseteq \mathrm{Cn}^{n}_{\mathcal{A}}(T) \supseteq \mathrm{Cn}^{n}_{\mathsf{M}}(T) = (\mathrm{Fm}^{n}_{\Sigma} \cap \bigcap \mathcal{U})$, where $\mathcal{U} \triangleq \{g^{-1}[D^{\mathcal{B}}] \supseteq T \mid \mathcal{B} \in \mathsf{M}, g \in \hom(\mathfrak{Fm}^{n}_{\Sigma}, \mathfrak{B})\}$ is both non-empty, for $T \neq \mathrm{Fm}^{n}_{\Sigma}$, and finite, for n as well as both M and all members of it are so. Let us prove, by contradiction, that $T \in \mathcal{U}$. For suppose $T \notin \mathcal{U}$. Take any enumeration $\overline{U}$ of $\mathcal{U}$. Then, for each $i \in m \triangleq |\mathcal{U}| \in (\omega \setminus 1)$, we have $T \subsetneq U_i$, in which case $U_i \nsubseteq T$, and so there is some $\varphi_i \in (U_i \setminus T) \neq \varnothing$. In this way, as (2.8) with $i \in 2$ are true in M, while $\mathcal{A}$ is $\veebar$-disjunctive, we get $(\veebar\bar{\varphi}) \in ((\mathrm{Fm}_\Sigma^n \cap \bigcap \mathcal{U}) \setminus T) = \varnothing$. This contradiction implies that $T \in \mathcal{U}$, in which case there are some $\mathcal{B} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}_\Sigma^n, \mathfrak{B})$ such that $T = g^{-1}[D^{\mathcal{B}}]$, and so $g \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$. Then, $E \triangleq (\mathrm{img}\, g)$ forms a subalgebra of $\mathfrak{B}$, in which case $\mathcal{E} \triangleq (\mathcal{B} \upharpoonright E) \in \mathbf{S}(\mathsf{M})$, and so $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{E})$. In particular, $\mathcal{E}$ is consistent, for $\mathcal{D}$ is so. Thus, $\mathcal{E} \in \mathbf{S}_*(\mathsf{M})$, $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{E})$ and $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{A})$, as required. □

Corollary 3.14. *Let C be a Σ-logic. [Suppose it is defined by a finite class M of finite Σ-matrices, in which* (2.8) *with $i \in 2$ are true (in particular, which are $\veebar$-disjunctive).] Then, (i)⇔(ii)⇔(iii)[⇔(iv)], where:*

(i) *C is purely-inferential;*
(ii) *C has a truth-empty model;*
(iii) *C has a one-valued truth-empty model;*
(iv) $\mathbf{S}_*(\mathsf{M})$ *has a truth-empty member.*

Proof. First, (ii)⇒(i) is immediate. The converse is by the fact that, by the structurality of C, $\langle \mathfrak{Fm}_\Sigma^\omega, C(\varnothing) \rangle$ is a model of C.

Next, (ii) is a particular case of (iii). Conversely, let $\mathcal{A} \in \mathrm{Mod}(C)$ be truth-empty. Then, $\chi^{\mathcal{A}}$ is singular, in which case $\theta^{\mathcal{A}} = A^2 \in \mathrm{Con}(\mathfrak{A})$, and so, by (2.20), $(\mathcal{A}/\theta^{\mathcal{A}}) \in \mathrm{Mod}(C)$ is both one-valued and truth-empty.

[Finally, (iv)⇒(ii) is by (2.20). Conversely, (iii)⇒(iv) is by Lemma 3.13 and the $\veebar$-disjunctivity of truth-empty Σ-matrices.] □

Theorem 3.15. *Let $\mathcal{A}$ be a finite Σ-matrix with unary unitary equality determinant Υ, C the logic of $\mathcal{A}$ and $\mathcal{B}$ a $\veebar$-disjunctive model of C. Suppose* (2.8) *with $i \in 2$ are true in $\mathcal{A}$ (in particular, it is $\veebar$-disjunctive). Then,* $\hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A}) \neq \varnothing$.

Proof. Consider any $F \in \wp_{\omega\setminus 1}(B)$. Then, by (2.20) and Remark 2.4(ii), the submatrix $\mathcal{B}_F$ of $\mathcal{B}$ generated by F is a finitely-generated $\veebar$-disjunctive model of C. Therefore, by Lemmas 2.5, 3.13, Remark 2.3(ii) and Theorem 3.10, there is some $h_F \in \hom_{\mathrm{S}}(\mathcal{B}_F, \mathcal{A})$. Now, consider any $G \in \wp_{\omega\setminus 1}(B)$ including F, in which case $B_F \subseteq B_G \subseteq B$, and any $a \in B_F$. Then, for each $\upsilon \in \Upsilon$, $(D^{\mathcal{A}} \ni \upsilon^{\mathfrak{A}}(h_F(a)) = h_F(\upsilon^{\mathfrak{B}_F}(a))) \Leftrightarrow (\upsilon^{\mathfrak{B}}(a) = \upsilon^{\mathfrak{B}_F}(a) \in D^{\mathcal{B}_F}) \Leftrightarrow (\upsilon^{\mathfrak{B}_G}(a) = \upsilon^{\mathfrak{B}}(a) \in D^{\mathcal{B}}) \Leftrightarrow (\upsilon^{\mathfrak{B}_G}(a) \in D^{\mathcal{B}_G}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h_G(a)) = h_G(\upsilon^{\mathfrak{B}_G}(a)) \in D^{\mathcal{A}})$, in which case $h_F(a) = h_G(a)$, and so $h_F \subseteq h_G$. Therefore, $\mathcal{H} \triangleq \{h_F \mid F \in \wp_{\omega\setminus 1}(B)\}$ is an upward-directed (for $\wp_{\omega\setminus 1}(B)$ is so) subset of the inductive set of all subalgebras of $\mathfrak{B} \times \mathfrak{A}$ (uniquely determined by, and so identified with their carriers). Hence, $h \triangleq \bigcup \mathcal{H}$ forms a subalgebra of $\mathfrak{B} \times \mathfrak{A}$. And what is more, $B = \bigcup \wp_{2\setminus 1}(B) \subseteq \bigcup \wp_{\omega\setminus 1}(B) \subseteq \bigcup \{B_F \mid F \in \wp_{\omega\setminus 1}(B)\} \subseteq B$, in which case $(\mathrm{dom}\, h) = \bigcup \{\mathrm{dom}\, f \mid f \in \mathcal{H}\} = \bigcup \{B_F \mid F \in \wp_{\omega\setminus 1}(B)\} = B$, while, for all $F, G \in \wp_{\omega\setminus 1}(B)$, $H \triangleq (F \cup G) \in \wp_{\omega\setminus 1}(B)$, in which case, for every $a \in (B_F \cap B_G)$, $h_F(a) = h_H(a) = h_G(a)$, and so h is a function, whereas $(\mathrm{img}\, h) = \bigcup \{\mathrm{img}\, f \mid f \in \mathcal{H}\} \subseteq A$, and so $h : B \to A$. In this way, $h \in \hom(\mathfrak{B}, \mathfrak{A})$. Finally, consider any $a \in B$, in which case $F \triangleq \{a\} \in \wp_{\omega\setminus 1}(B)$, and so $(a \in D^{\mathcal{B}}) \Leftrightarrow (a \in D^{\mathcal{B}_F}) \Leftrightarrow (D^{\mathcal{A}} \ni h_F(a) = h(a))$. Thus, $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. □

3.3. **Implicativity.** Fix any set A, any closure operator C over A and any $\iota : A^2 \to A$, in which case we put $\delta_\iota(a, b) : A^2 \to A$, $\langle a, b \rangle \mapsto \iota(\iota(a, b), b)$.

Next, C is said to *have Abstract Deduction Theorem (ADT) with respect to ι*, provided, for all $a \in X \subseteq A$ and all $b \in C(X)$, it holds that $\iota(a, b) \in C(X \setminus \{a\})$. Then, C is said to be *weakly ι-implicative*, provided it has ADT with respect to ι and

$$b \in C(\{a, \iota(a, b)\}), \tag{3.8}$$

for all $a, b \in A$. Likewise, C is said to be *(strongly) ι-implicative*, whenever it is weakly so and

$$\delta_\iota(\iota(a,b),a) \in C(\varnothing), \tag{3.9}$$

for all $a, b \in A$.

Lemma 3.16. *Suppose C is ι-implicative. Then, it is δ_ι-disjunctive.*

Proof. With using Lemma 3.12. Consider, any $(X \cup \{a,b\}) \subseteq A$. Then, (3.4) is by ADT w.r.t. ι. Next, (3.5) is by (3.8) and (3.9). Further, by (3.8) and ADT w.r.t. ι, we have $\iota(\iota(a,b),a) \in C(\{\iota(b,a), \delta_\iota(a,b)\})$, in which case, by (3.8) and (3.9), we get $a \in C(\{\iota(b,a), \delta_\iota(a,b)\})$, and so, by ADT w.r.t. ι, we eventually get (3.6). Finally, consider any $c \in C(X \cup \{b\})$. Then, by (3.8) and ADT w.r.t. ι, we have $\iota(b,a) \in C(X \cup \{\iota(c,a)\})$, in which case, by (3.8), we get $a \in C(X \cup \{\delta_\iota(b,a), \iota(c,a)\})$, and so, by ADT w.r.t. ι, we eventually get $\delta_\iota(c,a) \in C(X \cup \{\delta_\iota(b,a)\})$. Thus, (3.7) holds, as required. □

3.3.1. *Implicative calculi versus implicative logics.*

Lemma 3.17. *Let C' be a finitary Σ-logic and C'' a 1-extension of C'. Suppose C' has DT with respect to $\sqsupset$, while (2.11) is satisfied in C''. Then, C'' is an extension of C'.*

Proof. By induction on any $n \in \omega$, we prove that C'' is an n-extension of C'. For consider any $X \in \wp_n(\mathrm{Fm}^\omega_\Sigma)$, in which case $n \neq 0$, and any $\psi \in C'(X)$. Then, in case $X = \varnothing$, we have $X \in \wp_1(\mathrm{Fm}^\omega_\Sigma)$, and so $\psi \in C'(X) \subseteq C''(X)$, for C'' is a 1-extension of C'. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_{n-1}(\mathrm{Fm}^\omega_\Sigma)$, and so, by DT with respect to $\sqsupset$, that C' has, and the induction hypothesis, we have $(\phi \sqsupset \psi) \in C'(Y) \subseteq C''(Y)$. Therefore, by (2.11)$[x_0/\phi, x_1/\psi]$ satisfied in C'', in view of its structurality, we eventually get $\psi \in C''(Y \cup \{\phi\}) = C''(X)$. Hence, since $\omega = (\bigcup \omega)$, we eventually conclude that C'' is an ω-extension of C', and so an extension of C', for this is finitary. □

By $\mathfrak{I}^{[\mathrm{PL}]}_\sqsupset$ we denote the Σ-calculus constituted by (2.11), (2.13) and (2.14) [as well as (2.15)].

Lemma 3.18 (cf. Theorem 2.5 of [13])**.** *Let $\mathcal{A}$ be an axiomatic Σ-calculus, C' the Σ-logic axiomatized by $\mathfrak{I}_\sqsupset \cup \mathcal{A}$ and $\mathfrak{A}$ a Σ-algebra. Then, $\mathrm{Fg}^{\mathfrak{A}}_{C'}$ has ADT with respect to $\sqsupset^{\mathfrak{A}}$.*

Proof. Consider any $a \in X \subseteq A$ and any $b \in \mathrm{Fg}^{\mathfrak{A}}_{C'}(X)$, in which case there is some $(\mathfrak{I}_\sqsupset \cup \mathcal{A})$-derivation $\bar{c}$ of b from X over $\mathfrak{A}$. Then, by induction on any $i \in (\mathrm{dom}\,\bar{c})$, with using the derivability of (2.12) in $\mathfrak{I}_\sqsupset$ and Herbrand's method (cf., e.g., the proof of Proposition 1.8 of [10]), it is routine checking that $(a \sqsupset^{\mathfrak{A}} c_i) \in \mathrm{Fg}^{\mathfrak{A}}_{C'}(X \setminus \{a\})$. In this way, the fact that $b \in (\mathrm{img}\,\bar{c})$ completes the argument. □

Corollary 3.19. *Finitary weakly $\sqsupset$-implicative Σ-logics are exactly axiomatic extensions of the Σ-logic axiomatized by $\mathfrak{I}_\sqsupset$.*

Proof. Let C' be a finitary $\sqsupset$-implicative Σ-logic and C'' the Σ-logic axiomatized by $\mathfrak{I}_\sqsupset \cup C'(\varnothing)$. Then, C' is an extension of C''. Conversely, C'' is a 1-extension of C', and so, by Lemma 3.17, is an extension of C'. In this way, Lemma 3.18 with $\mathfrak{A} = \mathfrak{Fm}^\omega_\Sigma$ completes the argument. □

After all, combining Lemma 3.18 and Corollary 3.19, we immediately get:

Corollary 3.20. *Let C' be a finitary [weakly] $\sqsupset$-implicative Σ-logic and $\mathfrak{A}$ a Σ-algebra. Then, $\mathrm{Fg}^{\mathfrak{A}}_{C'}$ is [weakly] $\sqsupset^{\mathfrak{A}}$-implicative.*

4. Self-extensional logics versus simple matrices

Theorem 4.1. *Let* M *be a class of simple* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$*,* $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$*,* $\alpha \triangleq (1 \cup (\omega \cap \bigcup\{|A| \mid \mathcal{A} \in \mathsf{M}\})) \in \wp_{\infty\setminus 1}(\omega)$ *and* C *the logic of* M*. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) $\equiv^\omega_C \subseteq \theta^\omega_\mathsf{K}$*;*
(iii) $\equiv^\omega_C = \theta^\omega_\mathsf{K}$*;*
(iv) *for all distinct* $a, b \in F^\alpha_\mathsf{V}$*, there are some* $\mathcal{A} \in \mathsf{M}$ *and some* $h \in \hom(\mathfrak{F}^\alpha_\mathsf{V}, \mathfrak{A})$ *such that* $\chi^\mathcal{A}(h(a)) \neq \chi^\mathcal{A}(h(b))$*;*
(v) *there is some class* C *of* Σ*-algebras such that* $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$ *and, for each* $\mathfrak{A} \in \mathsf{C}$ *and all distinct* $a, b \in A$*, there are some* $\mathcal{B} \in \mathsf{M}$ *and some* $h \in \hom(\mathfrak{A}, \mathfrak{B})$ *such that* $\chi^\mathcal{B}(h(a)) \neq \chi^\mathcal{B}(h(b))$*;*
(vi) *there is some* $\mathsf{S} \subseteq \mathrm{Mod}(C)$ *such that* $\mathsf{K} \subseteq \mathbf{V}(\pi_0[\mathsf{S}])$ *and, for each* $\mathcal{A} \in \mathsf{S}$*, it holds that* $(A^2 \cap \bigcap\{\theta^\mathcal{B} \mid \mathcal{B} \in \mathsf{S}, \mathfrak{B} = \mathfrak{A}\}) \subseteq \Delta_A$*.*

Proof. First, (i/ii)⇒(ii/iii) is by Corollary 3.2/Lemma 3.5, respectively.

Next, assume (iii) holds. Then, $\theta^\beta \triangleq \equiv^\beta_C = \theta^\beta_\mathsf{K} = \theta^\beta_\mathsf{V} \in \mathrm{Con}(\mathfrak{Fm}^\beta_\Sigma)$, for all $\beta \in \wp_{\infty\setminus 1}(\omega)$. In particular (when $\beta = \omega$), (i) holds. Furthermore, consider any distinct $a, b \in F^\alpha_\mathsf{V}$. Then, there are some $\phi, \psi \in \mathrm{Fm}^\alpha_\Sigma$ such that $\nu_{\theta^\alpha}(\phi) = a \neq b = \nu_{\theta^\alpha}(\phi)$, in which case, by (2.19), $\mathrm{Cn}^\alpha_\mathsf{M}(\phi) \neq \mathrm{Cn}^\alpha_\mathsf{M}(\psi)$, and so there are some $\mathcal{A} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$ such that $\chi^\mathcal{A}(g(\phi)) \neq \chi^\mathcal{A}(g(\phi))$. In that case, $\theta^\alpha \subseteq (\ker g)$, and so, by the Homomorphism Theorem, $h \triangleq (g \circ \nu^{-1}_{\theta^\alpha}) \in \hom(\mathfrak{F}^\alpha_\mathsf{V}, \mathfrak{A})$. Then, $h(a/b) = g(\phi/\psi)$, in which case $\chi^\mathcal{A}(h(a)) \neq \chi^\mathcal{A}(h(b))$, and so (iv) holds.

Further, assume (iv) holds. Let $\mathsf{C} \triangleq \{\mathfrak{F}^\alpha_\mathsf{V}\}$. Consider any $\mathfrak{A} \in \mathsf{K}$ and the following complementary cases:

- $|A| \leqslant \alpha$.
  Let $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$ extend any surjection from V_α onto A, in which case it is surjective, while $\theta \triangleq \theta^\alpha_\mathsf{V} = \theta^\alpha_\mathsf{K} \subseteq (\ker h)$, and so, by the Homomorphism Theorem, $g \triangleq (h \circ \nu^{-1}_\theta) \in \hom(\mathfrak{F}^\alpha_\mathsf{V}, \mathfrak{A})$ is surjective. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^\alpha_\mathsf{V})$.
- $|A| \nleqslant \alpha$.
  Then, $\alpha = \omega$. Consider any Σ-identity $\phi \approx \psi$ true in $\mathfrak{F}^\omega_\mathsf{V}$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$, in which case, we have $\theta \triangleq \theta^\omega_\mathsf{V} = \theta^\omega_\mathsf{K} \subseteq (\ker h)$, and so, since $\nu_\theta \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{F}^\omega_\mathsf{V})$, we get $\langle \phi, \psi \rangle \in (\ker \nu_\theta) \subseteq (\ker h)$. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^\alpha_\mathsf{V})$.

Thus, $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$, and so (v) holds.

Now, assume (v) holds. Let C' be the class of all non-one-element members of C and $\mathsf{S} \triangleq \{\langle \mathfrak{A}, h^{-1}[D^\mathcal{B}] \rangle \mid \mathfrak{A} \in \mathsf{C}', \mathcal{B} \in \mathsf{M}, h \in \hom(\mathfrak{A}, \mathfrak{B})\}$. Then, for all $\mathfrak{A} \in \mathsf{C}'$, each $\mathcal{B} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{A}, \mathfrak{B})$, h is a strict homomorphism from $\mathcal{C} \triangleq \langle \mathfrak{A}, h^{-1}[D^\mathcal{B}] \rangle$ to $\mathcal{B}$, in which case, by (2.20), $\mathcal{C} \in \mathrm{Mod}(C)$, and so $\mathsf{S} \subseteq \mathrm{Mod}(C)$, while $\chi^\mathcal{C} = (\chi^\mathcal{B} \circ h)$, whereas $\pi_0[\mathsf{S}] = \mathsf{C}'$ generates the variety $\mathbf{V}(\mathsf{C})$. In this way, (vi) holds.

Finally, assume (vi) holds. Consider any $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$ such that $\phi \equiv^\omega_C \psi$, any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$. Then, for each $\mathcal{B} \in \mathsf{S}$ with $\mathfrak{B} = \mathfrak{A}$, $h(\phi)\ \theta^\mathcal{B}\ h(\psi)$, in which case $h(\phi) = h(\psi)$, and so $\mathfrak{A} \models (\phi \approx \psi)$. Thus, $\mathsf{K} \subseteq \mathbf{V}(\pi_0[\mathsf{S}]) \models (\phi \approx \psi)$, and so (ii) holds, as required. □

When both M and all members of it are finite, α is finite, in which case $\mathfrak{F}^\alpha_\mathsf{V}$ is finite and can be found effectively, and so, taking (2.20) and Remark 2.3[(iv)] into account, the item (iv) of Theorem 4.1 yields an effective procedure of checking the self-extensionality of any logic defined by a finite class of finite matrices. However, its computational complexity may be too large to count it *practically* applicable. For instance, in the unitary n-valued case, where $n \in \omega$, the upper limit n^{n^n} of

$|F^{\alpha}_{\mathsf{V}}|$ as well as the predetermined computational complexity $n^{n^{n^{n}}}$ of the procedure involved become too large even in the three-/four-valued case. And, though, in the two-valued case, this limit — 16 — as well as the respective complexity — $2^{16} = 65536$ — are reasonably acceptable, this is no longer matter in view of the following universal observation:

Example 4.2. Let $\mathcal{A}$ be a Σ-matrix. Suppose it is both false- and truth-singular (in particular, two-valued as well as both consistent and truth-non-empty [in particular, classical]), in which case $\theta^{\mathcal{A}} = \Delta_A$, for $\chi^{\mathcal{A}}$ is injective, and so $\mathcal{A}$ is simple. Then, by Theorems 3.7 and 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}\}$, the logic of $\mathcal{A}$ is self-extensional, its intrinsic variety being generated by $\mathfrak{A}$. Thus, by the self-extensionality of inferentially inconsistent logics, *any two-valued (in particular, classical) logic is self-extensional.* □

Nevertheless, the procedure involved is simplified much under certain conditions upon the basis of the item (v) of Theorem 4.1.

4.1. **Self-extensional conjunctive disjunctive logics.**

Lemma 4.3. *Let C be a [finitary $\bar{\wedge}$-conjunctive] Σ-logic and $\mathcal{A}$ a [truth-non-empty $\bar{\wedge}$-conjunctive] Σ-matrix. Then, $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$ if[f] $\mathcal{A} \in \mathrm{Mod}(C)$.*

Proof. The "if" part is trivial. [Conversely, assume $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$. Consider any $\varphi \in C(\varnothing)$, in which case $V \triangleq \mathrm{Var}(\varphi) \in \wp_{\omega}(V_{\omega})$, and so $(V_{\omega} \setminus V) \neq \varnothing$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Take any $v \in (V_{\omega} \setminus V) \neq \varnothing$ and any $a \in D^{\mathcal{A}} \neq \varnothing$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h{\restriction}(V_{\omega} \setminus \{v\})) \cup [v/a]$. Then, $\varphi \in C(v)$, in which case, as $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$ and $g(v) = a \in D^{\mathcal{A}}$, we have $h(\varphi) = g(\varphi) \in D^{\mathcal{A}}$, and so we get $\mathcal{A} \in \mathrm{Mod}_{2}(C)$. By induction on any $n \in \omega$, let us prove that $\mathcal{A} \in \mathrm{Mod}_{n}(C)$. For consider any $X \in \wp_{n}(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case $n \neq 0$. In case $|X| \in 2$, $X \in \wp_{2}(\mathrm{Fm}^{\omega}_{\Sigma})$, and so $C(X) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}(X)$, for $\mathcal{A} \in \mathrm{Mod}_{2}(C)$. Otherwise, $|X| \geqslant 2$, in which case there are some distinct $\phi, \psi \in X$, and so $Y \triangleq ((X \setminus \{\phi, \psi\}) \cup \{\phi \bar{\wedge} \psi\}) \in \wp_{n-1}(\mathrm{Fm}^{\omega}_{\Sigma})$. Then, by the induction hypothesis and the $\bar{\wedge}$-conjunctivity of both C and $\mathcal{A}$, we get $C(X) = C(Y) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}(Y) = \mathrm{Cn}^{\omega}_{\mathcal{A}}(X)$. Thus, $\mathcal{A} \in \mathrm{Mod}_{\omega}(C)$, for $\omega = (\bigcup \omega)$, and so $\mathcal{A} \in \mathrm{Mod}(C)$, for C is finitary.] □

Remark 4.4. Let C be a $\bar{\wedge}$-conjunctive or/and $\underline{\vee}$-disjunctive Σ-logic and $\phi \approx \psi$ a semi-lattice/"distributive lattice" identity for $\bar{\wedge}$ or/and $\underline{\vee}$, respectively. Then, $\phi \equiv^{\omega}_{C} \psi$. □

Theorem 4.5. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$, $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$ and C the logic of M. Suppose C is finitary (in particular, both M and all members of it are finite) and $\bar{\wedge}$-conjunctive (that is, all members of M are so) [as well as $\underline{\vee}$-disjunctive (in particular, all members of M are so)]. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\phi \approx (\phi \bar{\wedge} \psi)))$, while semi-lattice [resp., distributive lattice] identities for $\bar{\wedge}$ [and $\underline{\vee}$] are true in K;*
(iii) *every truth-non-empty $\bar{\wedge}$-conjunctive [consistent $\underline{\vee}$- disjunctive] Σ-matrix with underlying algebra in V is a model of C, while semi-lattice [resp., distributive lattice] identities for $\bar{\wedge}$ [and $\underline{\vee}$] are true in V;*
(iv) *every truth-non-empty $\bar{\wedge}$-conjunctive [consistent $\underline{\vee}$- disjunctive] Σ-matrix with underlying algebra in K is a model of C, while semi-lattice [resp., distributive lattice] identities for $\bar{\wedge}$ [and $\underline{\vee}$] are true in K.*

Proof. First, (i)⇒(ii) is by Theorem 4.1(i)⇒(iii), Remark 4.4 and the $\bar{\wedge}$-conjuctivity of C. Next, (ii)⇒(iii) is by Lemma 4.3. Further, (iv) is a particular case of (iii). Finally, (iv)⇒(i) is by Theorem 4.1(vi)⇒(i) with S, being the class of all truth-non-empty $\bar{\wedge}$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrices with underlying algebra in K, and the semilattice identities for $\bar{\wedge}$ [as well as the Prime Ideal Theorem for distributive lattices]. (More precisely, consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$, in which case, by the semilattice identities (more specifically, the commutativity one) for $\bar{\wedge}$, $a_i \neq (a_i \bar{\wedge}^{\mathfrak{A}} a_{1-i})$, for some $i \in 2$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, \{b \in A \mid a_i = (a_i \bar{\wedge}^{\mathfrak{A}} b)\}\rangle \in \mathsf{S}$ [resp., by the Prime Ideal Theorem, there is some $\mathcal{B} \in \mathsf{S}$] such that $\mathfrak{B} = \mathfrak{A}$ and $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$.) □

Theorem 4.6. *Let M be a finite class of finite hereditarily simple $\bar{\wedge}$-conjunctive $\veebar$-disjunctive Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, C is self-extensional iff, for each $\mathfrak{A} \in \mathsf{K}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some non-singular $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$.*

Proof. The "if" part is by Theorem 4.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, by Theorem 4.5(i)⇒(iv), $\mathfrak{A}$ is a distributive $(\bar{\wedge}, \veebar)$-lattice, in which case, by the commutativity identity for $\bar{\wedge}$, $a_i \neq (a_i \bar{\wedge}^{\mathfrak{A}} a_{1-i})$, for some $i \in 2$, and so, by the Prime Ideal Theorem, there is some $\bar{\wedge}$-conjunctive $\veebar$-disjunctive Σ-matrix $\mathcal{D}$ with $\mathfrak{D} = \mathfrak{A}$ such that $a_i \in D^{\mathcal{D}} \not\ni a_{1-i}$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, and so is a model of C. Hence, by Lemmas 2.5, 3.13 and Remark 2.3(ii), there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{A}, \mathfrak{B})$, in which case $h(a_i) \in D^{\mathcal{B}} \not\ni h(a_{1-i})$, and so $\chi^{\mathcal{B}}(h(a_i)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(a_{1-i}))$, in which case $h(a_i) \neq h(a_{1-i})$, and so h is not singular, as required. □

4.2. **Self-extensional implicative logics.**

Lemma 4.7. *Let C be a Σ-logic, $\mathcal{A} \in \mathrm{Mod}^*(C)$ and $\phi, \psi \in C(\varnothing)$. Suppose C is self-extensional. Then, $\mathfrak{A} \models (\phi \approx \psi)$.*

Proof. In that case, $\phi \equiv^{\omega}_{C} \psi$, and so Corollary 3.3 completes the argument. □

Lemma 4.8. *Let C be a Σ-logic, $\mathcal{A} \in \mathrm{Mod}^*(C)$, $a \in A$ and $\mathcal{B} \triangleq \langle \mathfrak{A}, \{a \sqsupset^{\mathfrak{A}} a\}\rangle$. Suppose C is finitary, self-extensional and weakly $\sqsupset$-implicattive. Then, $(a \sqsupset^{\mathfrak{A}} a) \sqsupset^{\mathfrak{A}} b) = b$, for all $b \in A$, in which case $\mathcal{B} \in \mathrm{Mod}(C)$, and so $D^{\mathcal{B}} = \mathrm{Fg}^{\mathfrak{A}}_{C}(\varnothing)$.*

Proof. Let $\varphi \in C(\varnothing)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $V \triangleq \mathrm{Var}(\phi) \in \wp_{\omega}(V_{\omega})$, in which case $(V_{\omega} \setminus V) \neq \varnothing$, and so there is some $v \in (V_{\omega} \setminus V)$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h \restriction (V_{\omega} \setminus \{v\})) \cup [v/a]$. Then, as, by (2.12), $(v \sqsupset v) \in C(\varnothing)$, by Lemma 4.7, we have $h(\varphi) = g(\varphi) = g(v \sqsupset v) = (a \sqsupset^{\mathfrak{A}} a) \in D^{\mathcal{B}}$, and so $\mathcal{B} \in \mathrm{Mod}_1(C)$. Moreover, as, by (2.12), $(x_0 \sqsupset x_0) \in C(\varnothing)$, by (2.13) and (2.11), we have $((x_0 \sqsupset x_0) \sqsupset x_1) \equiv^{\omega}_{C} x_1$, in which case, by Corollary 3.3, we get $(a \sqsupset^{\mathfrak{A}} a) \sqsupset^{\mathfrak{A}} b) = b$, for all $b \in A$, and so (2.11) is true in $\mathcal{B}$. In this way, (2.12) and Lemma 3.17 complete the argument. □

Theorem 4.9. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Suppose C is finitary (in particular, both M and all members of it are finite) and $\sqsupset$-implicative (in particular, all members of M are so). Then, C is self-extensional iff, for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\psi \approx (\phi \uplus_{\sqsupset} \psi)))$, while both (2.3) and (2.4) as well as semi-lattice identities for $\uplus_{\sqsupset}$ are true in K.*

Proof. The "if" part is by Theorem 4.1(ii)⇒(i) and semi-lattice identities (more specifically, the commutativity one) for $\uplus_{\sqsupset}$. Conversely, by Lemma 3.16, C is $\uplus_{\sqsupset}$-disjunctive. In this way, Theorem 4.1(i)⇒(iii), Remark 4.4, (2.12), Lemma 4.8 and the $\uplus_{\sqsupset}$-disjunctivity of C complete the argument. □

Now, we are in a position to prove the following "implicative" analogue of Theorem 4.6:

Theorem 4.10. *Let* M *be a finite class of finite hereditarily simple* $\sqsupset$*-implicative* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ *and* C *the logic of* M*. Then,* C *is self-extensional iff, for each* $\mathfrak{A} \in \mathsf{K}$ *and all distinct* $a, b \in A$*, there are some* $\mathcal{B} \in \mathsf{M}$ *and some non-singular* $h \in \hom(\mathfrak{A}, \mathfrak{B})$ *such that* $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$*.*

Proof. The "if" part is by Theorem 4.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any distinct $a, b \in A$. Then, by Theorem 4.9, $\mathfrak{A}$ is a $\uplus_{\sqsupset}$-semi-lattice satisfying (2.4), in which case, by the commutativity identity for $\uplus_{\sqsupset}$, without loss of generality, $b \neq (a \uplus^{\mathfrak{A}}_{\sqsupset} b)$, and so $b \notin \mathrm{Fg}^{\mathfrak{A}}_{C}(a)$, for, otherwise, by Corollary 3.20 and Lemma 4.8, we would have $(a \sqsupset^{\mathfrak{A}} b) \in \mathrm{Fg}^{\mathfrak{A}}_{C}(\varnothing) = \{a \sqsupset^{\mathfrak{A}} a\}$, in which case we would get $(a \sqsupset^{\mathfrak{A}} b) = (a \sqsupset^{\mathfrak{A}} a)$, and so, by (2.4), we would eventually get $(a \uplus^{\mathfrak{A}}_{\sqsupset} b) = ((a \sqsupset^{\mathfrak{A}} b) \sqsupset^{\mathfrak{A}} b) = ((a \sqsupset^{\mathfrak{A}} a) \sqsupset^{\mathfrak{A}} b) = b$. Therefore, $\mathfrak{F} \triangleq \{F \in \pi_1[\mathrm{Mod}(C, \mathfrak{A})] \mid b \notin F \ni a\} \ni \mathrm{Fg}^{\mathfrak{A}}_{C}(a)$, being inductive, for $\pi_1[\mathrm{Mod}(C, \mathfrak{A})]$ is so, is non-empty, and so has a maximal element G, in view of Zorn Lemma. We are going to prove that $\mathcal{D} \triangleq \langle \mathfrak{A}, G \rangle$ is $\uplus_{\sqsupset}$-disjunctive. For consider any $c, d \in A$. Then, in case $(c/d) \in G$, by Corollary 3.20, Lemma 3.16 and (3.3)/(3.4), we have $(c \uplus^{\mathfrak{A}}_{\sqsupset} d) \in G$. Conversely, assume $c, d \in (A \setminus G)$. Let us prove, by contradiction, that $(c \uplus^{\mathfrak{A}}_{\sqsupset} d) \notin G$. For suppose $(c \uplus^{\mathfrak{A}}_{\sqsupset} d) \in G$. Then, by the maximality of G, we have $b \in (\mathrm{Fg}^{\mathfrak{A}}_{C}(G \cup \{c\}) \cap \mathrm{Fg}^{\mathfrak{A}}_{C}(G \cup \{d\})$, in which case, as $\mathrm{Fg}^{\mathfrak{A}}_{C}$, being $\sqsupset^{\mathfrak{A}}$-implicative, by Corollary 3.20, is $\veebar^{\mathfrak{A}}_{\sqsupset}$-disjunctive, by Lemma 3.16, we get $b \in \mathrm{Fg}^{\mathfrak{A}}_{C}(G \cup \{c \veebar^{\mathfrak{A}}_{\sqsupset} d\}) = \mathrm{Fg}^{\mathfrak{A}}_{C}(G) = G$, and so this contradiction shows that $(c \uplus^{\mathfrak{A}}_{\sqsupset} d) \notin G$. Thus, $\mathcal{D}$ is $\uplus_{\sqsupset}$-disjunctive. Hence, by Lemmas 2.5, 3.13 and Remark 2.3(ii), there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{A}, \mathfrak{B})$, in which case, as $b \notin G = D^{\mathcal{D}} \ni a$, we have $h(a) \in D^{\mathcal{B}} \not\ni h(b)$, and so we get $\chi^{\mathcal{B}}(h(a)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(b))$, in which case $h(a) \neq h(b)$, so h is not singular. □

4.3. **Common consequences.** The effective procedure of verifying the self-extensionality of an n-valued implicative/"both disjunctive and conjunctive" logic, where $n \in \omega$, resulted from Theorem 4.6/4.10 has the computational complexity n^n that is quite acceptable for (3|4)-valued logics. And what is more, it provides a quite useful heuristic tool of doing it, manual applications of which (suppressing the factor n^n at all) are presented below. First, we have:

Corollary 4.11. *Let* $n \in (\omega \setminus 3)$*,* $\mathcal{A}$ *a finite hereditarily simple* $\sqsupset$*-implicative/"both* $\barwedge$*-conjunctive and* $\veebar$*-disjunctive"* Σ*-matrix and* C *the logic of* $\mathcal{A}$*. Suppose every non-singular endomorphism of* $\mathfrak{A}$ *is diagonal. Then, the logic of* $\mathcal{A}$ *is not self-extensional.*

Proof. By contradiction. For suppose C is self-extensional. Then, as $n \in (\omega \setminus 3)$, $\mathcal{A}$ is either false- or truth-non-singular, in which case $\chi^{\mathcal{A}}$ is not injective, and so there are some distinct $a, b \in A$ such that $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(b)$. On the other hand, by Theorem 4.6/4.10, there is some non-singular $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, in which case $h = \Delta_A$, and so $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b)) = \chi^{\mathcal{A}}(b)$. This contradiction completes the argument. □

4.3.1. *Self-extensionality versus algebraizability.* We start from proving the following "implicative" analogue of Lemma 11 of [19] being interesting in its own right within the context of Universal Algebra:

Lemma 4.12. *Let* $\mathcal{A}$ *be an* $\sqsupset$*-implicative* Σ*-matrix with [finite] unary unitary equality determinant* Υ*. Suppose* $\mathfrak{A}$ *is an* $\sqsupset$*-implicative inner semi-lattice. Then,* $\mho^{\sqsupset}_{\Upsilon} \triangleq \{((\gamma(x_i) \sqsupset \gamma(x_{1-i})) \uplus_{\sqsupset} (\delta(x_{2+j}) \sqsupset \delta(x_{2+1-j}))) \approx (x_0 \sqsupset x_0) \mid i, j \in 2, \gamma, \delta \in \Upsilon\}$ *is a [finite] disjunctive system for* $\mathfrak{A}$*.*

Proof. Consider any $\bar{a} \in A^4$. Let $h \in \hom(\mathfrak{Fm}^4_\Sigma, \mathfrak{A})$ extend $[x_i/a_i]_{i\in 4}$.

First, assume $(a_0 = a_1)|(a_2 = a_3)$. Then, for each $(\gamma|\delta) \in \Upsilon$ and every $(i|j) \in 2$, $(\gamma|\delta)^{\mathfrak{A}}(a_{i|(2+j)}) = (\gamma|\delta)^{\mathfrak{A}}(a_{(1-i)|(2+1-j)})$, in which case $((\gamma|\delta)^{\mathfrak{A}}(a_{i|(2+j)}) \sqsupset^{\mathfrak{A}} (\gamma|\delta)^{\mathfrak{A}}(a_{(1-i)|(2+1-j)})) = \flat^{\mathfrak{A}}_{\uplus_\sqsupset}$, and so, for each $(\delta|\gamma) \in \Upsilon$ and every $(j|i) \in 2$, $((\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \uplus^{\mathfrak{A}}_\sqsupset (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))) = \flat^{\mathfrak{A}}_{\uplus_\sqsupset} = (a_0 \sqsupset^{\mathfrak{A}} a_0)$. Thus, $\mathfrak{A} \models (\bigwedge \mho^{\sqsupset}_{\Upsilon})[h]$.

Conversely, assume both $a_0 \neq a_1$ and $a_2 \neq a_3$. Then, there are some $\gamma, \delta \in \Upsilon$ and some $i, j \in 2$ such that both $\gamma^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}} \not\ni \gamma^{\mathfrak{A}}(a_{1-i})$ and $\delta^{\mathfrak{A}}(a_{2+j}) \in D^{\mathcal{A}} \not\ni \delta^{\mathfrak{A}}(a_{2+1-j})$, in which case, by the $\sqsupset$-implicativity of $\mathcal{A}$, $(\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \notin D^{\mathcal{A}} \not\ni (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))$, and so, by the $\uplus_\sqsupset$-disjunctivity of $\mathcal{A}$, $((\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \uplus^{\mathfrak{A}}_\sqsupset (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))) \notin D^{\mathcal{A}}$. On the other hand, by the $\sqsupset$-implicativity of $\mathcal{A}$, $(a_0 \sqsupset^{\mathfrak{A}} a_0) \in D^{\mathcal{A}}$. Hence, $((\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \uplus^{\mathfrak{A}}_\sqsupset (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))) \neq (a_0 \sqsupset^{\mathfrak{A}} a_0)$. Thus, $\mathfrak{A} \not\models (\bigwedge \mho^{\sqsupset}_{\Upsilon})[h]$. ☐

According to [19], given any $m, n \in \omega$, a *(Σ-)equational $\vdash^m_n$-(sequent)definition for* a Σ-matrix $\mathcal{A}$ is any $\Omega \in \wp_\omega(\mathrm{Eq}^{m+n}_\Sigma)$ such that, for all $\bar{a} \in A^m$ and all $\bar{b} \in A^n$, it holds that $(((\mathrm{img}\, a) \subseteq D^{\mathcal{A}}) \Rightarrow (((\mathrm{img}\, b) \cap D^{\mathcal{A}}) \neq \varnothing)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \Omega)[x_i/a_i; x_{m+j}/b_j]_{i\in m; j\in n})$. (Equational $\vdash^{0/1}_1$-definitions are also referred to as *equational "truth definitions"/implications*, respectively/, according to Appendix A of [21].) Some kinds of equational sequent definitions are actually equivalent for implicative matrices, in view of the following compound immediate observation:

Remark 4.13. Given a[n $\sqsupset$-implicative] Σ-matrix $\mathcal{A}$, (i[-v]) does [resp., do] hold, where:

(i) given any equational $\vdash^2_2$-definition Ω for $\mathcal{A}$, $\Omega[x_{(2\cdot i)+j}/x_i]_{i,j\in 2}$ is an equational implication for $\mathcal{A}$ (cf. Theorems 10 and 12(ii)⇒(iii) of [21]);

(ii) given any equational implication Ω for $\mathcal{A}$, $\Omega[x_0/(x_0 \sqsupset x_0), x_1/x_0]$ is an equational truth definition for $\mathcal{A}$;

(iii) given any equational truth definition Ω for $\mathcal{A}$, the following hold:
a) $\Omega[x_0/(x_0 \sqsupset x_1)]$ is an equational implication for $\mathcal{A}$;
b) $\Omega[x_0/(x_0 \sqsupset (x_1 \sqsupset (x_2 \uplus_\sqsupset x_3)))]$ is an equational $\vdash^2_2$-definition for $\mathcal{A}$;

(iv) given any unary binary equality determinant ε (in particular, $\varepsilon = \varepsilon_\Upsilon$, where Υ is a unary unitary equality determinant) for $\mathcal{A}$, $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon\}$ is an axiomatic binary equality determinant for $\mathcal{A}$;

(v) in case $\mathcal{A}$ is truth-singular, $\{x_0 \approx (x_0 \sqsupset x_0\}$ is an equational truth definition for it. ☐

In this way, taking Theorems 10, 12(i)⇔(ii) and 13 of [19] as well as Remark 4.13 into account, a "both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive"/$\sqsupset$-implicative consistent truth-non-empty finite Σ-matrix $\mathcal{M}$ with unary unitary equality determinant has an equational implication iff a multi-conclusion two-side sequent calculus $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ (cf. [18] as well as the paragraph -2 on p. 294 of [19] for more detail)/" (or the equivalent — in the sense of [16] — logic of $\mathcal{M}$)" is algebraizable — in the sense of [16]. In this connection, by Lemma 9 and Theorem[s] 10 [and 14(ii)⇒(i)] of [19] [as well as Lemma 4.12/"11 of [19]"], we have

Lemma 4.14 (cf. Theorem[s] 14 [and 15] of [19] [for the "lattice conjunctive disjunctive" case])**.** *Let $\mathcal{A}$ be a finite consistent truth-non-empty [$\sqsupset$-implicative/"both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive"] Σ-matrix with unary unitary equality determinant. [Suppose $\mathfrak{A}$ is an/a "$\sqsupset$-implicative inner semi-lattice"/$(\bar{\wedge}, \veebar)$-lattice, respectively.] Then, $\mathcal{A}$ has an equational implication [if and] only if every non-singular partial endomorphism of $\mathfrak{A}$ is diagonal.*

As a consequence, by Theorem 3.10, Corollary 4.11 and Lemma 4.14, we immediately get the following universal negative result:

Corollary 4.15. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ an n-valued consistent truth-non-empty $\sqsupset$-implicative/"both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive" Σ-matrix with unary unitary equality determinant and C the logic of $\mathcal{A}$. Suppose $\mathcal{A}$ has an equational implication. Then, C is not self-extensional.*

The converse does not, generally speaking, hold — even in the "lattice conjunctive disjunctive" case (cf. Example 5.22), though does hold within the framework of three-valued paraconsistent/paracomplete logics with subclassical negation as well as "lattice conjunction and disjunction"|"implicative inner semi-lattice implication" (cf. Corollary 5.37|5.47, respectively). In view of Theorem 10 and Lemma 8 of [19], Example 4.2 and the self-extensionality of inferentially inconsistent logics, the reservations "$n \in (\omega \setminus 3)$" and "$n$-valued consistent truth-non-empty" cannot be omitted in the formulation of Corollary 4.15.

Example 4.16 (Łukasiewicz' finitely-valued logics; cf. [9])**.** Let $n \in (\omega \setminus 3)$, $\Sigma \triangleq (\Sigma_{+,\sim} \cup \{\supset\})$ with binary $\supset$ (implication) and $\mathcal{A}$ the Σ-matrix with $(\mathfrak{A}{\restriction}\Sigma_{+}) \triangleq \mathfrak{D}_n$, $D^{\mathcal{A}} \triangleq \{1\}$, $\sim^{\mathfrak{A}} \triangleq (1-a)$ and $(a \supset^{\mathfrak{A}} b) \triangleq \min(1, 1-a+b)$, for all $a, b \in A$, in which case $\mathcal{A}$ is both consistent, truth-non-empty, $\wedge$-conjunctive and $\veebar$-disjunctive as well as, by Example 7 of [19], is implicative, and so, by Remark 4.13(v),(iii)**a)**, has an equational implication (cf. Example 7 of [19]) and, by Example 3 of [18], a unary unitary equality determinant. Hence, by Corollary 4.15, the logic of $\mathcal{A}$ is not self-extensional. □

Example 4.17. In view of Remarks 1 and 2 of [19], Lemma 4.14 and Corollaries 4.11 and 4.15, arbitrary three-valued expansions of both the *logic of paradox LP* [12] and Kleene's three-valued logic K_3 [7] are not self-extensional, because the former has the equational implication $(x_0 \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge x_1)$, discovered in [15], while the latter has the same underlying algebra. Likewise, in view of "Proposition 5.7 of [21]"/"both Lemma 4.1 of [13] and Remark 4.13(iii)**a)**" as well as Corollary 4.15, arbitrary three-valued expansions of P^1/HZ [22]/[6] are not self-extensional, for they have an equational implication/"truth definition", respectively. □

Another generic applications of our universal elaboration are discussed in the next section.

5. Applications and examples

5.1. **Four-valued expansions of Belnap's four-valued logic.** Here, it is supposed that $\Sigma \supseteq \Sigma_{+,\sim[,01]}$. Fix a Σ-matrix $\mathcal{A}$ with $(\mathfrak{A}{\restriction}\Sigma_{+,\sim[,01]}) \triangleq \mathfrak{DM}_{4[,01]}$ and $D^{\mathcal{A}} \triangleq (2^2 \cap \pi_0^{-1}[\{1\}])$, Then, both $\mathcal{A}$ and $\partial(\mathcal{A}) \triangleq \langle \mathfrak{A}, 2^2 \cap \pi_1^{-1}[\{1\}]\rangle$ are both $\wedge$-conjunctive and $\vee$-disjunctive, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for them (cf. Example 2 of [18]), so they as well as their submatrices are hereditarily simple (cf. Theorem 3.10), while:

$$(\theta^{\mathcal{A}} \cap \theta^{\partial(\mathcal{A})}) = \Delta_A, \tag{5.1}$$

$$D^{\partial(\mathcal{A})} = (\sim^{\mathfrak{A}})^{-1}[A \setminus D^{\mathcal{A}}]. \tag{5.2}$$

Let C be the logic of $\mathcal{A}$. Then, as $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A}{\restriction}\Sigma_{+,\sim[,01]})$ defines [the bounded version/expansion of] Belnap's four-valued logic $B_{4[,01]}$ [3] (cf. [14]), C is a four-valued expansion of $B_{4[,01]}$. This exhaust all four-valued expansions of $B_{4[,01]}$, $\mathcal{A}$ being uniquely determined by C, as we show below, marking the framework of the present subsection:

Lemma 5.1. *Any $\Sigma_{+,\sim[,01]}$-matrix $\mathcal{B}$ defines $B_{4[,01]}$ and is four-valued iff it is isomorphic to $\mathcal{DM}_{4[,01]}$, in which case it is simple.*

Proof. The "if" part is by (2.20) and the fact that $|2^2| = 4$. Conversely, assume $B_{4[,01]}$ is defined by $\mathcal{B}$, while this is four-valued. Then, by (2.20) and Remark 2.3[(iv)], $\mathcal{D} \triangleq (\mathcal{B}/\theta)$, where $\theta \triangleq \partial(\mathcal{B})$, is a simple $\Sigma_{+,\sim[,01]}$-matrix defining $B_{4[,01]}$. Hence, by Theorem 3.7, $\mathfrak{D}$ and $\mathfrak{DM}_{4[,01]}$ generate the same (intrinsic) variety (of $B_{4[,01]}$), in which case they satisfy same identities, and so the former is a [bounded] De Morgan lattice, for the latter is so. In particular,

$$((x_0 \wedge \sim x_0) \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge \sim x_0), \tag{5.3}$$

not being true in the latter under $[x_i/\langle i, 1-i\rangle]_{i\in 2}$, is not true in the former, in which case $\mathfrak{D}{\restriction}\Sigma_+$ is not a chain, and so there are some $a, b \in D$ such that $D \ni (c|d) \triangleq (a(\wedge|\vee)^{\mathfrak{D}} b) \notin \{a, b\}$. Then, $a \neq b$, in which case $c \neq d$, and so $D = \{a, b, c, d\}$, for $|D| \leqslant |B| = 4$. Therefore, $|D| = 4 \nleqslant 3$, in which case θ is diagonal, and so ν_θ is an isomorphism from $\mathcal{B}$ onto $\mathcal{D}$. Hence, $c|d$ is a zero|unit of $\mathfrak{D}{\restriction}\Sigma_+$, in which case $[(c|d) = (\perp|\top)^{\mathfrak{D}}$, while], by (2.6)|(2.7), $\sim^{\mathfrak{D}}(c|d) = (d|c)$, and so, by (2.5), $\sim^{\mathfrak{D}}(a/b) \notin \{c, d\}$. On the other hand, if $\sim^{\mathfrak{A}}(a/b)$ was equal to b/a, then, by (2.5), $\sim^{\mathfrak{A}}(b/a)$ was equal to a/b, in which case $e(\wedge|\vee)^{\mathfrak{D}} \sim^{\mathfrak{D}} e$ would be equal to $c|d$, for all $e \in D$, and so (5.3) would be true in $\mathfrak{D}$. Thus, $\sim^{\mathfrak{D}}(a/b) = (a/b)$. And what is more, $\mathcal{D}$ is both consistent, truth-non-empty and $\wedge$-conjunctive, for $\mathcal{DM}_{4[,01]}$ is so, that is, $B_{4[,01]}$ is both inferentially consistent and $\wedge$-conjunctive. Hence, $c \notin D^{\mathcal{D}} \ni d$, in which case $\{a, b\} \nsubseteq D^{\mathcal{D}}$, and so $(\{a, b\} \cap D^{\mathcal{D}}) \neq \varnothing$, for, otherwise, $D^{\mathcal{D}}$ would be equal to $\{d\}$, in which case $\mathcal{D}$ would be non-$\sim$-paraconsistent, and so would be $B_{4[,01]}$, contrary to the fact that (2.16) is not true in $\mathcal{DM}_{4[,01]}$ under $[x_i/\langle 1-i, i\rangle]_{i\in 2}$. Therefore, $D^{\mathcal{D}} = \{d, e\}$, for some $e \in \{a, b\}$, in which case the mapping $g : 2^2 \to D$, given by:

$$\begin{aligned} g(11) &\triangleq d, \\ g(00) &\triangleq c, \\ g(10) &\triangleq e, \\ g(01) &\triangleq f, \end{aligned}$$

where f is the unique element of $\{a, b\} \setminus \{e\}$, is an isomorphism from $\mathcal{DM}_{4[,01]}$ onto $\mathcal{D}$, and so $g^{-1} \circ \nu_\theta$ is that from $\mathcal{B}$ onto $\mathcal{DM}_{4[,01]}$. Finally, the simplicity of the latter and Remark 2.3[(iii)] complete the argument. □

Theorem 5.2. *Any four-valued Σ-expansion C' of $B_{4[,01]}$ is defined by a unique Σ-expansion of $\mathcal{DM}_{4[,01]}$.*

Proof. Let $\mathcal{A}'$ be a four-valued Σ-matrix defining C'. Then, $\mathcal{A}'{\restriction}\Sigma_{+,\sim[,01]}$ is a four-valued $\Sigma_{+,\sim[,01]}$-matrix defining $B_{4[,01]}$, in which case, by Lemma 5.1, there is some isomorphism e from $\mathcal{A}'{\restriction}\Sigma_{+,\sim[,01]}$ onto $\mathcal{DM}_{4[,01]}$, and so e is an isomorphism from $\mathcal{A}'$ onto the Σ-expansion $\mathcal{A}'' \triangleq \langle e[\mathfrak{A}'], 2^2 \cap \pi_0^{-1}[\{1\}]\rangle$ of $\mathcal{DM}_{4[,01]}$. Hence, by (2.20), C' is defined by $\mathcal{A}''$, being both finite and $\veebar$-disjunctive as well as having a unary unitary equality determinant. Finally, let $\mathcal{A}'''$ be any more Σ-expansion of $\mathcal{DM}_{4[,01]}$ defining C', in which case it is a $\vee$-disjunctive model of C', and so, by Theorem 3.15, there is some $h \in \hom_{\mathrm{S}}(\mathcal{A}''', \mathcal{A}'')$. Then, $h \in \hom_{\mathrm{S}}(\mathcal{DM}_{4[,01]}, \mathcal{DM}_{4[,01]})$, in which case, by Lemma 3.11, h is diagonal, and so $\mathcal{A}''' = \mathcal{A}''$, as required. □

Let $\mu : 2^2 \to 2^2, \langle i, j\rangle \mapsto \langle j, i\rangle$ and $\sqsubseteq \triangleq \{\langle ij, kl\rangle \in (2^2)^2 \mid i \leqslant k, l \leqslant j\}$, those n-ary operations on 2^2, where $n \in \omega$, which "commute with μ"/"are monotonic with respect to $\sqsubseteq$", being said to be *specular/regular*, respectively. Then, $\mathfrak{A}$ is said to be *specular/regular*, whenever its primary operations are so, in which case secondary ones are so as well. (Clearly, $\mathfrak{DM}_{4[,01]}$ is both specular and regular.) Then:

$$D^{\partial(\mathcal{A})} = \mu^{-1}[D^{\mathcal{A}}]. \tag{5.4}$$

Theorem 5.3. *The following are equivalent:*

(i) *C is self-extensional;*
(ii) *$\mathfrak{A}$ is specular;*
(iii) *$\partial(\mathcal{A})$ is isomorphic to $\mathcal{A}$;*
(iv) *C is defined by $\partial(\mathcal{A})$;*
(v) *$\partial(\mathcal{A}) \in \mathrm{Mod}(C)$;*
(vi) *C has PWC with respect to $\sim$.*

Proof. First, assume (i) holds. Then, by Theorem 4.6, there is some non-singular $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(11)) \neq \chi^{\mathcal{A}}(h(10))$, in which case $B \triangleq (\operatorname{img} h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, and so $\Delta_2 \subseteq B$. Hence, $\mathfrak{A}[\restriction B]$ is a $(\wedge, \vee)$-lattice with zero/unit $\langle 0/1, 0/1\rangle$, in which case, by Lemma 2.1, $(h\restriction\Delta_2)$ is diagonal, and so $h(10) \notin D^{\mathcal{A}}$, for $h(11) = (11) \in D^{\mathcal{A}}$. On the other hand, for all $a \in A$, it holds that $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \notin \Delta_2)$. Therefore, $h(10) = (01)$. Moreover, if $h(01)$ was equal to 01 too, then we would have $(00) = h(00) = h((10) \wedge^{\mathfrak{A}} (01)) = ((01) \wedge^{\mathfrak{A}} (01)) = (01)$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = \mu$, so (ii) holds.

Next, (ii)⇒(iii) is by (5.4) and the bijectivity of $\mu : A \to A$, while (iii)⇒(iv) is by (2.20), whereas (v) is a particular case of (iv). Further, (i)⇒(vi) is by:

Claim 5.4. *Any self-extensional extension C' of C has PWC with respect to $\sim$.*

Proof. In that case, C' is $\wedge$-conjunctive and satisfies (2.8) with $i = 1$, for C is and does so. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $\psi \in C'(\phi)$, in which case both $\sim(\phi \wedge \psi) \equiv_C (\sim\phi \vee \sim\psi)$, in view of (2.6), true in $\mathfrak{A}$, and Lemma 3.5, and $(\phi \wedge \psi) \equiv_{C'} \phi$, in view the $\wedge$-conjunctivity of C', and so, by (2.8) with $i = 1$ and the self-extensionality of C', $\sim\phi \equiv_{C'} (\sim\phi \vee \sim\psi) \in C'(\sim\psi)$, as required. □

Now, assume (vi) holds. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C(\phi)$, in which case $\sim\phi \in C(\sim\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h(\phi) \in D^{\partial(\mathcal{A})}$, in which case, by (5.2), $h(\sim\phi) \notin D^{\mathcal{A}}$, and so $h(\sim\psi) \notin D^{\mathcal{A}}$, that is, $h(\psi) \in D^{\partial(\mathcal{A})}$. Thus, $\partial(\mathcal{A})$, being both truth-non-empty and $\bar{\wedge}$-conjunctive, is a $(2 \setminus 1)$-model of C, and so, by Lemma 4.3, (v) holds.

Finally, (v)⇒(i) is by (5.1) and Theorem 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \partial(\mathcal{A})\}$. □

This positively covers $B_{4[,01]}$ as regular instances. And what is more, in case $\Sigma = \Sigma_{\simeq,+[,01]} \triangleq (\Sigma_{\sim,+[,01]} \cup \{\neg\})$ with unary $\neg$ (classical — viz., Boolean — negation) and $\neg^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-i, 1-j\rangle$, Theorem 5.3 equally covers the logic $CB_{4[,01]} \triangleq C$ of the $(\neg x_0 \vee x_1)$-implicative $\mathcal{DMB}_{4[,01]} \triangleq \mathcal{A}$ with non-regular — because of $\neg^{\mathfrak{A}}$ — underlying algebra, introduced in [17]. Below, we disclose a *unique* (up to term-wise definitional equivalence) status of these three self-extensional instances.

Lemma 5.5. *Suppose $\mathfrak{A}$ is specular. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$. In particular, C is $\sim$-subclassical, whenever it is self-extensional.*

Proof. By contradiction. For suppose there are some $f \in \Sigma$ of arity $n \in \omega$ and some $\bar{a} \in \Delta_2^n$ such that $f^{\mathfrak{A}}(\bar{a}) \notin \Delta_2$. Then, $f^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{A}}(\mu \circ \bar{a}) = \mu(f^{\mathfrak{A}}(\bar{a})) \neq f^{\mathfrak{A}}(\bar{a})$. This contradiction, Theorem 5.3(i)⇒(ii) and (2.20) complete the argument. □

Corollary 5.6. *Suppose C is self-extensional. Then, the following are equivalent:*

(i) *C has a theorem;*
(ii) *$\top^{\mathfrak{DM}_{4,01}}$ is term-wise definable in $\mathfrak{A}$;*
(iii) *$\bot^{\mathfrak{DM}_{4,01}}$ is term-wise definable in $\mathfrak{A}$;*
(iv) *$\{01\}$ does not form a subalgebra of $\mathfrak{A}$;*

(v) $\{10\}$ *does not form a subalgebra of* $\mathfrak{A}$.

Proof. Then, by Theorem 5.3(i)⇒(ii), $\mu \in \hom(\mathfrak{A}, \mathfrak{A})$. First, (i,iv) are particular cases of (ii), for $(01) \neq \top^{\mathfrak{DM}_{4,01}} = (11) \in D^{\mathcal{A}}$. Next, (ii)⇔(iii) is by the equalities $\sim^{\mathfrak{A}}(\bot^{\mathfrak{DM}_{4,01}}/\top^{\mathfrak{DM}_{4,01}}) = (\top^{\mathfrak{DM}_{4,01}}/\bot^{\mathfrak{DM}_{4,01}})$. Likewise, (iv)⇔(v) is by the equalities $\mu[\{01/10\}] = \{10/01\}$. Further, (i)⇒(ii) is by Lemmas 4.7 and 5.5. Finally, assume (iv) holds. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(01) \neq (01)$, in which case, by the injectivity of μ, we have $(10) = \mu(01) \neq \mu(\varphi^{\mathfrak{A}}(01)) = \varphi^{\mathfrak{A}}(\mu(01)) = \varphi^{\mathfrak{A}}(10)$, and so, by Lemma 5.5, we get $(x_0 \vee (\varphi \vee \sim\varphi)) \in C(\varnothing)$. Thus, (i) holds. □

Corollary 5.7. *Suppose* C *is self-extensional, and* $\mathcal{A}$ *is* $\sqsupset$*-implicative. Then,* $\neg^{\mathfrak{DMB}_4}$ *is term-wise definable in* $\mathfrak{A}$.

Proof. Then, by (2.12), true in $\mathcal{A}$, and Corollary 5.6(i)⇒(iii), $\bot^{\mathfrak{DM}_{4,01}} \notin D^{\mathcal{A}}$ is term-wise definable in $\mathfrak{A}$ by some $\tau \in \mathrm{Fm}^1_\Sigma$, and so $\mathcal{A}$ is $-$-negative, where $-x_0 \triangleq (x_0 \sqsupset \tau)$. Moreover, by Theorem 5.3, $\mathfrak{A}$ is specular, in which case, by Lemma 5.5, Δ_2 forms a subalgebra of $\mathfrak{A}$, and so $(-^{\mathfrak{A}}{\restriction}\Delta_2) = (\neg^{\mathfrak{DMB}_4}{\restriction}\Delta_2)$. On the other hand, if $-^{\mathfrak{A}}(10) \notin D^{\mathcal{A}}$ was equal to 00, then, as $(01) \notin D^{\mathcal{A}}$, we would have $D^{\mathcal{A}} \ni -^{\mathfrak{A}}(01) = -^{\mathfrak{A}}(\mu(10)) = \mu(-^{\mathfrak{A}}(10)) = \mu(00) = (00) \notin D^{\mathcal{A}}$. Therefore, $-^{\mathfrak{A}}(10) = (01)$, in which case $(10) = \mu(01) = \mu(-^{\mathfrak{A}}(10)) = -^{\mathfrak{A}}\mu(10) = -^{\mathfrak{A}}(01)$, and so $-^{\mathfrak{A}} = \neg^{\mathfrak{DMB}_4}$. □

5.1.1. *Specular functional completeness.* As usual, *Boolean algebras* are supposed to be of the signature $\Sigma^- \triangleq (\Sigma_{\simeq,+,01} \setminus \{\sim\})$, the ordinary one over 2 being denoted by $\mathfrak{B}_2$.

Lemma 5.8. *Let* $n \in \omega$ *and* $f : 2^n \to 2$. *[Suppose* f *is monotonic with respect to* $\leqslant$ *(and* $f(n \times \{i\}) = i$, *for each* $i \in 2$, *in which case* $n > 0$*).] Then, there is some* $\vartheta \in \mathrm{Fm}^n_{\Sigma^-[\setminus\{\neg(,\bot,\top)\}]}$ *such that* $g = \vartheta^{\mathfrak{B}_2}$.

Proof. Then, by the functional completeness of $\mathfrak{B}_2$, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^-}$ such that $g = \vartheta^{\mathfrak{B}_2}(\notin \{2^n \times \{i\} \mid i \in 2\})$, in which case, without loss of generality, one can assume that $\vartheta = (\wedge\langle\bar{\varphi}, \top\rangle)$, where, for each $m \in \ell \triangleq (\mathrm{dom}\,\bar{\varphi}) \in (\omega(\setminus 1))$, $\varphi_m = (\vee\langle(\neg \circ \bar{\phi}_m) * \bar{\psi}_m, \bot\rangle)$, for some $\bar{\phi}_m \in V_n^{k_m}$, some $\bar{\psi}_m \in V_n^{l_m}$ and some $k_m, l_m \in \omega$ such that $((\mathrm{img}\,\bar{\phi}_m) \cap (\mathrm{img}\,\bar{\psi}_m)) = \varnothing$. [Set $\zeta \triangleq (\wedge\langle\bar{\eta}, \top\rangle)$, where, for each $m \in (\mathrm{dom}\,\bar{\eta}) \triangleq \ell$, $\eta_m \triangleq (\vee\langle\bar{\psi}_m, \bot\rangle)$. Consider any $\bar{a} \in A^n$ and the following exhaustive cases:

(1) $g(\bar{a}) = 0$,
in which case we have $\zeta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} \leqslant \vartheta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 0$, and so we get $\zeta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 0$.

(2) $g(\bar{a}) = 1$,
in which case, for every $m \in \ell$, as $\bar{a} \leqslant \bar{b}_m \triangleq ((\bar{a}{\restriction}(n \setminus N_m)) \cup (N_m \times \{1\})) \in A^n$, where $N_m \triangleq \{j \in n \mid x_j \in (\mathrm{img}\,\bar{\phi}_m)\}$, by the monotonicity of g w.r.t. $\leqslant$, we have $1 = g(\bar{a}) \leqslant g(\bar{b}_m) = \vartheta^{\mathfrak{B}_2}[x_j/b_{m,j}]_{j\in n} \leqslant \varphi_m^{\mathfrak{B}_2}[x_j/b_{m,j}]_{j\in n} = \eta_m^{\mathfrak{B}_2}[x_j/a_j]_{j\in n}$, and so we get $\zeta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 1$.

Thus, $g = \zeta^{\mathfrak{B}_2}$. (And what is more, since, in that case, $\ell > 0$ and $l_m > 0$, for each $m \in \ell$, we also have $g = \xi^{\mathfrak{B}_2}$, where $\xi \triangleq (\wedge\bar{\upsilon})$, whereas, for each $m \in (\mathrm{dom}\,\bar{\upsilon}) \triangleq \ell$, $\upsilon_m \triangleq (\vee\bar{\psi}_m)$.)] This completes the argument. □

Theorem 5.9. *Let* $\Sigma = \Sigma_{\simeq,+,01}$, $n \in (\omega(\setminus 1))$ *and* $f : A^n \to A$. *Then,* f *is specular [and regular (as well as* $f(n \times \{a\}) = a$, *for all* $a \in (A \setminus \Delta_A)$*)] iff there is some* $\tau \in \mathrm{Fm}^n_{\Sigma[\setminus\{\neg(,\bot,\top)\}]}$ *such that* $f = \tau^{\mathfrak{A}}$.

Proof. The "if" part is immediate. Conversely, assume f is specular [and regular (as well as $f(n \times \{a\}) = a$, for all $a \in (A \setminus \Delta_A)$)]. Then,

$$g : 2^{2\cdot n} \to 2, \bar{a} \mapsto \pi_0(f(\langle\langle a_{2\cdot j}, 1 - a_{(2\cdot j)+1}\rangle\rangle_{j\in n}))$$

[is monotonic w.r.t. $\leqslant$ (and $g(n \times \{i\}) = i$, for each $i \in 2$)]. Therefore, by Lemma 5.8, there is some $\vartheta \in \mathrm{Fm}^{2\cdot n}_{\Sigma^-[\setminus\{\neg(,\bot,\top)\}]}$ such that $g = \vartheta^{\mathfrak{B}_2}$. Put

$$\tau \triangleq (\vartheta[x_{2\cdot j}/x_j, x_{(2\cdot j)+1}/{\sim} x_j]_{j\in n}) \in \mathrm{Fm}^n_{\Sigma[\setminus\{\neg(,\bot,\top)\}]}.$$

Consider any $\bar{c} \in A^n$. Then, since, for each $i \in 2$, we have $\pi_i \in \hom(\mathfrak{A}{\restriction}\Sigma^-, \mathfrak{B}_2)$, we get $\pi_0(\tau^{\mathfrak{A}}[x_j/c_j]_{j\in n}) = \vartheta^{\mathfrak{B}_2}[x_{2\cdot j}/\pi_0(c_j), x_{(2\cdot j)+1}/(1-\pi_1(c_j))]_{j\in n} = \pi_0(f(\bar{c}))$ and, likewise, as f is specular, $\pi_1(\tau^{\mathfrak{A}}[x_j/c_j]_{j\in n}) = \vartheta^{\mathfrak{B}_2}[x_{2\cdot j}/\pi_1(c_j), x_{(2\cdot j)+1}/(1-\pi_0(c_j))]_{j\in n} = \pi_0(f(\mu\circ\bar{c})) = \pi_0(\mu(f(\bar{c}))) = \pi_1(f(\bar{c}))$, as required. □

In this way, by Theorems 5.2, 5.3 and 5.9, $CB_{4[,01]}$ is the most expansive (up to term-wise definitional equivalence) self-extensional four-valued expansion of B_4. And what is more, combining Theorems 5.3 and 5.9 with Corollaries 5.6 and 5.7, we eventually get:

Corollary 5.10. *C is self-extensional, while $\mathcal{A}$ is implicative/"both $\mathfrak{A}$ is regular and C is [not] purely-inferential", iff C is term-wise definitionally equivalent to $CB_4/B_{4[,01]}$, respectively.*

5.1.2. *No-more-than-three-valued extensions.*

Lemma 5.11. *Let $n \in (4 \setminus 1)$. Then, any n-valued model/extension of C is $\vee$-disjunctive.*

Proof. Let $\mathcal{B}$ be an n-valued model of C, in which case, by (2.20) and Remark 2.3[(iv)], $\mathcal{D} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$, is an m-valued simple model of C, where $m \leqslant n \leqslant 3$, and so, by Corollary 3.6, $\mathfrak{D} \in \mathbf{V}(\mathfrak{A})$. Therefore, $\mathfrak{D}{\restriction}\Sigma_+$, being an m-element lattice, for $\mathfrak{A}{\restriction}\Sigma_+$ is a lattice, is a chain. Hence, $\mathcal{D}$, being $\wedge$-conjunctive, for C is so, is $\vee$-disjunctive, and so is $\mathcal{B}$, by Remark 2.4(ii), as required. □

Given any $i \in 2$, put $DM_{3,i} \triangleq (2^2 \setminus \{\langle i, 1-i\rangle\})$. Then, in case this forms a subalgebra of $\mathfrak{A}$ (such is the case, when, e.g., $\Sigma = \Sigma_{\sim,+[,01]}$), we set $(\mathcal{A}/\mathcal{DM})_{3,i/[,01]} \triangleq ((\mathcal{A}/\mathcal{DM})_{/4[,01]}{\restriction}DM_{3,i})$, the logic $(C/B)_{3,i/[,01]}$ of which is a both $\vee$-disjunctive and $\wedge$-connjunctive (for its defining matrix is so; cf. Remark 2.4(ii)) as well as inferentially consistent (for its defining matrix is both consistent and truth-non-empty) unitary three-valued both extension of $(C/B)_{4/[,01]}$, in view of (2.20), and expansion of $LP|K_3$, whenever $i = (0|1)$, in which case it is $\sim$-paraconsistent$|(\vee,\sim)$-paracomplete, and so is not $\sim$-classical.

Corollary 5.12. *Let $\mathcal{B}$ be a consistent truth-non-empty non-$\sim$-negative three-valued model of C and C' the logic of $\mathcal{B}$. Then, there is some $i \in 2$ such that $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is isomorphic to $\mathcal{A}_{3,i}$, and so $C' = C_{3,i}$.*

Proof. Then, by Lemma 5.11, $\mathcal{B}$ is $\vee$-disjunctive. Hence, by Theorem 3.15, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\mathrm{img}\, h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A}{\restriction}D)$. Therefore, if h was not injective, then $\mathcal{D}$ would be either one-valued, in which case it would be either inconsistent or truth-empty, and so would be $\mathcal{B}$, or two-valued, in which case D would be equal to Δ_2, and so, by Remark 2.4(ii), $\mathcal{B}$ would be $\sim$-negative, for $\mathcal{D}$ would be so. Thus, h is injective, in which case $|D| = 3$, and so $D = DM_{3,i}$, for some $i \in 2$. In this way, (2.20) completes the argument. □

Likewise, we have:

Corollary 5.13. *Let $\mathcal{B}$ be a consistent truth-non-empty two-valued model of C and C' the logic of $\mathcal{B}$. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is isomorphic to $\mathcal{A}{\restriction}\Delta_2$, in which case it is $\sim$-classical, and so is C'.*

Proof. Then, by Lemma 5.11, $\mathcal{B}$ is $\vee$-disjunctive. Hence, by Theorem 3.15, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \restriction D)$. Therefore, if h was not injective, then $\mathcal{D}$ would be one-valued, in which case it would be either inconsistent or truth-empty, and so would be $\mathcal{B}$. Thus, h is injective, in which case $|D| = 2$, and so $D = \Delta_2$. In this way, Remark 2.4(ii) completes the argument. □

And what is more, we also have:

Lemma 5.14. *Let $\mathcal{B}$ be a $\sim$-negative model of C and C' the logic of $\mathcal{B}$. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is a strict surjective homomorphic counter-image of $\mathcal{A} \restriction \Delta_2$, an so C' is $\sim$-classical.*

Proof. Then, by the following auxiliary observation, $\mathcal{B}$ is $\vee$-disjunctive:

Claim 5.15. *Any $\sim$-negative $\mathcal{B} \in \mathrm{Mod}(C)$ is $\vee$-disjunctive.*

Proof. Then, by Remark 2.4(i)**a)**, $\mathcal{B}$, being $\wedge$-conjunctive, for C is so, is $\wedge^{\sim}$-disjunctive. On the other hand, as (2.5) and (2.7) are true in $\mathfrak{A}$, so is $(x_0 \vee x_1) \approx (x_0 \wedge^{\sim} x_1)$, in which case, by Lemma 3.5, $(x_0 \vee x_1) \equiv^{\omega}_{C} (x_0 \wedge^{\sim} x_1)$, and so $((a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}) \Leftrightarrow ((a(\wedge^{\sim})^{\mathfrak{B}} b) \in D^{\mathcal{B}})$, for all $a, b \in B$. Thus, $\mathcal{B}$, being $\wedge^{\sim}$-disjunctive, is equally $\vee$-disjunctive, as required. □

Hence, by Theorem 3.15, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \restriction D)$, and so, by Remark 2.4(ii), $\mathcal{D}$ is $\sim$-negative, for $\mathcal{B}$ is so. Therefore, $D = \Delta_2$. Finally, (2.20) completes the argument. □

By Corollary 5.13, Lemma 5.14 and (2.20), we immediately have:

Theorem 5.16. *The following are equivalent:*

(i) *C is $\sim$-subclassical;*
(ii) *C has a consistent truth-non-empty two-valued model;*
(iii) *C has a $\sim$-negative model;*
(iv) *Δ_2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A} \restriction \Delta_2$ is a $\sim$-classical model of C isomorphic to any consistent truth-non-empty two-valued (in particular, $\sim$-classical) model of C and being a strict surjective homomorphic image of any $\sim$-negative model of C, and so defines a unique inferentially consistent two-valued (in particular, $\sim$-classical) extension of C.*

Likewise, Examples 4.2, 4.17, Corollary 5.12, Lemma 5.14 and the self-extensionality of inferentially inconsistent logics then immediately yield:

Theorem 5.17. *Let C' be a three-valued extension of C. Then, the following are equivalent:*

(i) *C' is self-extensional;*
(ii) *C' is either inferentially inconsistent or $\sim$-classical;*
(iii) *for each $i \in 2$, if $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, then $C' \neq C_{3,i}$.*

In general, since $\mathcal{DM}_4 \restriction \{01\}$ is the only truth-empty submatrix of $\mathcal{DM}_4$, by Corollaries 3.14, 5.12, Theprem 5.16 and (2.20), we also have:

Theorem 5.18. *Let M be a non-empty class of consistent no-more-than-three-valued models of C, C' the logic of M, $n \in (4 \setminus 1)$ and $\mathsf{M}^{(*)\{,\sim|\not\sim\}}_{[n]\langle,0/1\rangle}$ the class of all (truth-non-empty) [n-valued] $\{\sim$-negative$|$non-$\sim$-negative$\}$ $\langle$false-/truth-singular$\rangle$ members of M. Then, C' is defined by $\{\mathcal{A} \restriction \{01\} \mid (\mathsf{M} \setminus \mathsf{M}^*) \neq \varnothing = \mathsf{M}^{*,\not\sim}_{3,1}\} \cup \{\mathcal{A} \restriction \Delta_2 \mid (\bigcup_{i \in 2} \mathsf{M}^{*,\not\sim}_{3,i}) = \varnothing \neq (\mathsf{M}^{\sim} \cup \mathsf{M}^{*}_{2})\} \cup \bigcup_{i \in 2} \{\mathcal{A}_{3,i} \mid \mathsf{M}^{*,\not\sim}_{3,i} \neq \varnothing\}$.*

In view of Theorem 5.17, any inferentially consistent non-$\sim$-classical *unitary* three-valued extension of C' is not self-extensional. Then, taking (2.18), Theorem 5.18, Remark 2.2 and Example 4.2 into account, for analyzing the "non-unitary" case it suffices to restrict our consideration by the following "double" one.

5.1.2.1. Double three-valued extension. Here, it is supposed that, for each $i \in 2$, $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.20), the logic $(C/B)_{3/[,01]}$ of $\{(\mathcal{A}/\mathcal{DM})_{3,0/[,01]}, (\mathcal{A}/\mathcal{DM})_{3,1/[,01]}\}$ is the $\vee$-disjunctive both $\sim$-paraconsistent (for $(\mathcal{A}/\mathcal{DM})_{3,0/[,01]}$ is so) — in particular, non-$\sim$-classical — and $(\vee,\sim)$-paracomplete (for $(\mathcal{A}/\mathcal{DM})_{3,1/[,01]}$ is so) proper extension of $C/B_{4[,01]}$ satisfying $\{x_0, \sim x_0\} \vdash (x_1 \vee \sim x_1)$, for this is not true in $\mathcal{A}/\mathcal{DM}_{4[,01]}$ under $[x_i/\langle 1-i, i\rangle]_{i\in 2}$, and so $\Delta_2 = (DM_{3,0} \cap DM_{3,1})$ forms a subalgebra of $\mathfrak{A}_{[3,0]}$, in which case $C_{[3]}$ is $\sim$-subclassical, in view of (2.20). Moreover, set $\partial(\mathcal{A}_{3,i}) \triangleq (\partial(\mathcal{A}) \upharpoonright DM_{3,i})$.

Theorem 5.19. *The following are equivalent:*

(i) *C_3 is self-extensional;*
(ii) *for each $i \in 2$, $(\mu \upharpoonright DM_{3,i}) \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$;*
(iii) *for some $i \in 2$, $(\mu \upharpoonright DM_{3,i}) \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$;*
(iv) *for each $i \in 2$, C_3 is defined by $\{\mathcal{A}_{3,i}, \partial(\mathcal{A}_{3,i})\}$;*
(v) *for some $i \in 2$, C_3 is defined by $\{\mathcal{A}_{3,i}, \partial(\mathcal{A}_{3,i})\}$;*
(vi) *for each $i \in 2$, $\partial(\mathcal{A}_{3,i}) \in \mathrm{Mod}(C_3)$;*
(vii) *for some $i \in 2$, $\partial(\mathcal{A}_{3,i}) \in \mathrm{Mod}(C_3)$;*
(viii) *$\mathfrak{A}_{3,0}$ and $\mathfrak{A}_{3,1}$ are isomorphic;*
(ix) *C_3 has PWC with respect to $\sim$;*
(x) *$\mathfrak{A}$ has a non-diagonal non-singular partial endomorphism.*

Proof. First, assume (i) holds. Consider any $i \in 2$. Then, as $DM_{3,i} \ni a \triangleq \langle 1-i, i\rangle \neq b \triangleq \langle 1-i, 1-i\rangle \in \Delta_2 \subseteq DM_{3,i}$, by Theorem 4.6, there are some $j \in 2$, some non-singular $h \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,j})$ such that $\chi^{\mathcal{A}_{3,j}}(h(a)) \neq \chi^{\mathcal{A}_{3,j}}(h(b))$, in which case $B \triangleq (\mathrm{img}\, h)$ forms a non-one-element subalgebra of $\mathfrak{A}_{3,j}$, and so $\Delta_2 \subseteq B$. Hence, $\mathfrak{A}_{3,i[-i+j]}[\upharpoonright B]$ is a $(\wedge,\vee)$-lattice with zero/unit $\langle 0/1, 0/1\rangle$, in which case, by Lemma 2.1, $(h \upharpoonright \Delta_2)$ is diagonal, and so $h(b) = b \in D^{\mathcal{A}_j}$. On the other hand, for all $c \in A$, it holds that $(\sim^{\mathfrak{A}} c = c) \Leftrightarrow (c \notin \Delta_2)$. Therefore, as $a \notin \Delta_2$, $h(a) \notin \Delta_2$, in which case $B \neq \Delta_2$, and so $B = DM_{3,j}$. Hence, if j was equal to i, we would have $h(a) = a$, in which case we would get $\chi^{\mathcal{A}_{3,j}}(h(a)) = \chi^{\mathcal{A}_{3,j}}(a) = (1-i) = \chi^{\mathcal{A}_{3,j}}(b) = \chi^{\mathcal{A}_{3,j}}(h(b))$, and so $j = (1-i)$, in which case $h(a) = \mu(a)$. Thus, $\hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i}) \ni h = (\mu \upharpoonright DM_{3,i})$, and so (ii) holds.

Next, (iii/v/vii) is a particular case of (ii/iv/vi), respectively, while (viii) is a particular case of (iii). Likewise, (vi/vii) is a particular case of (iv/v), while (ii/iii)$\Rightarrow$(iv/v) is by (2.20) and (5.4).

Further, assume (vii) holds. Then, as no false-/truth-singular Σ-matrix is isomorphic to any one not being so, while $\partial(\mathcal{A}_{3,i})$ is false-/truth-singular iff $\mathcal{A}_{3,i}$ is not so, by Remarks 2.3(ii), 2.4(ii) and Lemmas 2.5 and 3.13, we conclude that $\partial(\mathcal{A}_{3,i})$ is isomorphic to $\mathcal{A}_{3,1-i}$, and so (2.20) yields (v).

Now, assume (viii) holds. Let e be any isomorphism from $\mathfrak{A}_{3,0}$ onto $\mathfrak{A}_{3,1}$. Then, since these are both $(\wedge,\vee)$-lattices with zero/unit $\langle 0/1, 0/1\rangle$, by Lemma 2.1, $e \upharpoonright \Delta_2$ is diagonal. Moreover, for all $c \in A$, it holds that $(\sim^{\mathfrak{A}} c = c) \Leftrightarrow (c \notin \Delta_2)$. Therefore, $e(10) = (01)$, in which case $\hom(\mathfrak{A}_{3,0}, \mathfrak{A}_{3,1}) \ni e = (\mu \upharpoonright DM_{3,0})$, and so (iii) with $i = 0$ holds.

Furthermore, (v)$\Rightarrow$(i) is by Theorem 4.1(vi)$\Rightarrow$(i) with $\mathsf{S} = \mathsf{M} = \{\mathcal{A}_{3,i}, \partial(\mathcal{A}_{3,i})\}$ and (5.1), while (i)$\Rightarrow$(ix) is by Claim 5.4.

Conversely, assume (ix) holds. Consider any $i \in 2$, any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C_3(\phi)$, in which case $\sim\phi \in C_3(\sim\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A}_{3,i})$ such that $h(\phi) \in D^{\partial(\mathcal{A}_{3,i})}$, in which case, by (5.2), $h(\sim\phi) \notin D^{\mathcal{A}_{3,i}}$, and so $h(\sim\psi) \notin D^{\mathcal{A}_{3,i}}$, that is, $h(\psi) \in$

$D^{\partial(\mathcal{A}_{3,i})}$. Thus, $\partial(\mathcal{A}_{3,i})$ is a $(2\setminus 1)$-model of C. Moreover, by Remark 2.4(ii), it is $\bar{\wedge}$-conjunctive, for $\partial(\mathcal{A})$ is so, and so, by Lemma 4.3, (vi) holds.

Finally, (x) is a particular case of (iii). Conversely, assume (x) holds. Then, there are some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and some non-diagonal non-singular $h \in \hom(\mathfrak{B},\mathfrak{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, and so does $B = (\operatorname{dom} h)$. Hence, $\Delta_2 \subseteq (B \cap D)$. Therefore, both $\mathfrak{B}$ and $\mathfrak{D}$ are $(\wedge,\vee)$-lattices with zero/unit $\langle 0/1, 0/1\rangle$, in which case, as $h \in \hom(\mathfrak{B},\mathfrak{D})$ is surjective, by Lemma 2.1, $h{\restriction}\Delta_2$ is diagonal, and so there is some $i \in 2$ such that $DM_{3,i} \subseteq B$, while $h(\langle 1-i, i\rangle) \neq \langle 1-i, i\rangle$. On the other hand, for all $a \in A$, it holds that $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \notin \Delta_2)$, in which case $\sim^{\mathfrak{A}} h(\langle 1-i,i\rangle) = h(\sim^{\mathfrak{A}}\langle 1-i,i\rangle) = h(\langle 1-i,i\rangle)$, and so $h(\langle 1-i,i\rangle) = \langle i, 1-i\rangle$. In this way, $\hom(\mathfrak{A}_{3,i},\mathfrak{A}) \ni (h{\restriction}DM_{3,i}) = (\mu{\restriction}DM_{3,i})$, in which case $(\mu{\restriction}DM_{3,i}) \in \hom(\mathfrak{A}_{3,i},\mathfrak{A}_{3,1-i})$, and so (iii) holds, as required. □

First, by Lemma 4.14 and Theorem 5.19(i)⇔(x), we immediately have:

Corollary 5.20. *C_3 is self-extensional iff $\mathcal{A}$ has no equational implication.*

Then, by Corollaries 4.15 and 5.20, we also have:

Corollary 5.21. *C_3 is self-extensional, whenever C is so.*

On the other hand, the converse does not hold, as it follows from:

Example 5.22 (cf. Example 11 of [19])**.** Let $\Sigma \triangleq (\Sigma_{\sim,+[,01]} \cup \{\amalg\})$ with binary $\amalg$ and $\amalg^{\mathfrak{A}} \triangleq ((\vee^{\mathfrak{A}}{\restriction}(DM_{3,0}^2 \cup DM_{3,1}^2)) \cup \{\langle\langle 01,10\rangle,11\rangle,\langle\langle 10,01\rangle,00\rangle\})$. Then, $\mathfrak{A}$ is not specular, while $(\mu{\restriction}DM_{3,0}) \in \hom(\mathfrak{A}_{3,0},\mathfrak{A}_{3,1})$. Hence, by Theorems 5.3, 5.19 and Corollary 5.20, C_3 is self-extensional, while C is not so, whereas $\mathcal{A}$ has no equational implication. □

5.2. **Disjunctive three-valued logics with subclassical negation.** A Σ-matrix $\mathcal{A}$ is said to be $\sim$*-super-classical*, if $\mathcal{A}{\restriction}\{\sim\}$ has a $\sim$-classical submatrix, in which case $\mathcal{A}$ is both consistent and truth-non-empty, while, by (2.20), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$, and so we have the "if" part of the following preliminary marking the framework of the present subsection:

Theorem 5.23. *Let $\mathcal{A}$ be a Σ-matrix. [Suppose $|A| \leqslant 3$.] Then, $\sim$ is a subclassical negation for the logic of $\mathcal{A}$ if[f] $\mathcal{A}$ is $\sim$-super-classical.*

Proof. [Assume $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. First, by (2.21) with $m = 1$ and $n = 0$, there is some $a \in D^{\mathcal{A}}$ such that $\sim^{\mathfrak{A}} a \notin D^{\mathcal{A}}$. Likewise, by (2.21) with $m = 0$ and $n = 1$, there is some $b \in (A \setminus D^{\mathcal{A}})$ such that $\sim^{\mathfrak{A}} b \in D^{\mathcal{A}}$, in which case $a \neq b$, and so $|A| \neq 1$. Then, if $|A| = 2$, we have $A = \{a, b\}$, in which case $\mathcal{A}$ is $\sim$-classical, and so $\sim$-super-classical. Now, assume $|A| = 3$.

Claim 5.24. *Let $\mathcal{A}$ be a three-valued Σ-matrix, $\bar{a} \in A^2$ and $i \in 2$. Suppose $\sim$ is a subclassical negation for the logic of $\mathcal{A}$ and, for each $j \in 2$, $(a_j \in D^{\mathcal{A}}) \Leftrightarrow (\sim^{\mathfrak{A}} a_j \notin D^{\mathcal{A}}) \Leftrightarrow (a_{1-j} \notin D^{\mathcal{A}})$. Then, either $\sim^{\mathfrak{A}} a_i = a_{1-i}$ or $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$.*

Proof. By contradiction. For suppose both $\sim^{\mathfrak{A}} a_i \neq a_{1-i}$ and $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i \neq a_i$. Then, in case $a_i \in / \notin D^{\mathcal{A}}$, as $|A| = 3$, we have both $(D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}})) = \{a_i\}$, in which case $\sim^{\mathfrak{A}} a_{1-i} = a_i$, and $((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}) = \{a_{1-i}, \sim^{\mathfrak{A}} a_i\}$, respectively. Consider the following exhaustive cases:

- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_{1-i}$.
  Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$. This contradicts to (2.21) with $(n/m) = 0$ and $(m/n) = 3$, respectively.
- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = \sim^{\mathfrak{A}} a_i$.
  Then, for each $c \in ((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}})$, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} c = \sim^{\mathfrak{A}} a_i \notin / \in D^{\mathcal{A}}$. This contradicts to (2.21) with $(n/m) = 3$ and $(m/n) = 0$, respectively.

Thus, in any case, we come to a contradiction, as required. □

Finally, consider the following exhaustive cases:

- both $\sim^{\mathfrak{A}} a = b$ and $\sim^{\mathfrak{A}} b = a$.
  Then, $\{a,b\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{a,b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} a \neq b$.
  Then, by Claim 5.24, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, in which case $\{a, \sim^{\mathfrak{A}} a\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{a, \sim^{\mathfrak{A}} a\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} b \neq a$.
  Then, by Claim 5.24, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} b = b$, in which case $\{b, \sim^{\mathfrak{A}} b\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{b, \sim^{\mathfrak{A}} b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.] □

The following counterexample shows that the optional condition $|A| \leqslant 3$ is essential for the optional "only if" part of Theorem 5.23 to hold:

Example 5.25. Let $n \in \omega$ and $\mathcal{A}$ any Σ-matrix with $A \triangleq (n \cup (2 \times 2))$, $D^{\mathcal{A}} \triangleq \{\langle 1,0\rangle, \langle 1,1\rangle\}$, $\sim^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, (1-i+j) \mod 2\rangle$, for all $i,j \in 2$, and $\sim^{\mathfrak{A}} k \triangleq \langle 1,0\rangle$, for all $k \in n$. Then, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}\restriction\{\sim\}$, we have $(2\times 2) \subseteq B$, in which case $4 \leqslant |B|$, and so $\mathcal{A}$ is not $\sim$-super-classical, for $4 \nleqslant 2$. On the other hand, 2×2 forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $\mathcal{B} \triangleq (\mathcal{A}\restriction\{\sim\})\restriction(2\times 2)$ being $\sim$-negative, in which case $\chi^{\mathcal{A}}\restriction(2\times 2)$ is a surjective strict homomorphism from $\mathcal{B}$ onto the $\sim$-classical $\{\sim\}$-matrix $\mathcal{C}$ with $C \triangleq 2$, $D^{\mathcal{C}} \triangleq \{1\}$ and $\sim^{\mathfrak{C}} i \triangleq (1-i)$, for all $i \in 2$, and so, by (2.20), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. □

Let $\mathcal{A}$ be a fixed three-valued $\sim$-super-classical (in particular, both consistent and truth-non-empty) Σ-matrix and $\mathcal{B}$ a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$. Then, as $4 \nleqslant 3$, $\mathcal{A}$ is either false-singular, in which case the unique non-distinguished value $0_{\mathcal{A}}$ of $\mathcal{A}$ is that $0_{\mathcal{B}}$ of $\mathcal{B}$, so $1^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}} 0_{\mathcal{A}} = \sim^{\mathfrak{B}} 0_{\mathcal{B}} = 1_{\mathcal{B}}$, or truth-singular, in which case the unique distinguished value $1_{\mathcal{A}}$ of $\mathcal{A}$ is that $1_{\mathcal{B}}$ of $\mathcal{B}$, so $0^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}} 1_{\mathcal{A}} = \sim^{\mathfrak{B}} 1_{\mathcal{B}} = 0_{\mathcal{B}}$. Thus, in case $\mathcal{A}$ is false-/truth-singular, $B = 2^{\sim}_{\mathcal{A}} \triangleq \{0^{/\sim}_{\mathcal{A}}, 1^{\sim/}_{\mathcal{A}}\}$ is uniquely determined by $\mathcal{A}$ and $\sim$, the unique element of $A \setminus 2^{\sim}_{\mathcal{A}}$ being denoted by $(\frac{1}{2})^{\sim}_{\mathcal{A}}$. (The indexes $_{\mathcal{A}}$ and, especially, $^{\sim}$ are often omitted, unless any confusion is possible.) Strict homomorphisms from $\mathcal{A}$ to itself retain both 0 and 1, in which case surjective ones retain $\frac{1}{2}$, and so:

$$\hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{A}) \supseteq [=]\{\Delta_A\}, \tag{5.5}$$

the inclusion [not] being allowed to be proper (cf. Example 5.28 below).

From now on, unless otherwise specified, C is supposed to be the logic of $\mathcal{A}$. Then, C is $\bar{\wedge}$-conjunctive iff $\mathcal{A}$ is so. It appears that such does hold for both disjunctivity and implicativity too, as it ensues from the following two lemmas:

Lemma 5.26. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose [either] $\mathcal{B}$ is false-singular (in particular, $\sim$-classical) [or both $\mathcal{B}$ is $\sim$-super-classical and $|B| \leqslant 3$]. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive;*
(ii) *$\mathcal{B}$ is $\veebar$-disjunctive;*
(iii) (2.8) *with $i = 0$,* (2.9) *and* (2.10) *[as well as the* Resolution *rule:*

$$\{x_0 \veebar x_1, \sim x_0 \veebar x_1\} \vdash x_1] \tag{5.6}$$

are satisfied in C' (viz., true in $\mathcal{B}$);

(iv) (2.8) *with* $i = 0$, (2.9) *and* (2.10) *[as well as the* Modus ponens *rule for the* material implication $\sim x_0 \veebar x_1$*:*

$$\{x_0, \sim x_0 \veebar x_1\} \vdash x_1] \tag{5.7}$$

are satisfied in C' *(viz., true in* $\mathcal{B}$*).*

Proof. First, (ii)⇒(i) is immediate.

Next, assume (i) holds. Then, (2.8) with $i = 0$, (2.9) and (2.10) are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, not $\sim$-paraconsistent. Hence, $x_1 \in (C'(x_1) \cap C'(\{x_0, \sim x_0\})) = (C'(x_1) \cap C'(\{x_0 \veebar x_1, \sim x_0\})) = C'(\{x_0 \veebar x_1, \sim x_0 \veebar x_1\})$, so (5.6) is satisfied in C'.] Thus, (iii) holds.

Further, (iv) is a particular case of (iii) [for (5.7) is that of (5.6), in view of (2.8) with $i = 0$].

Finally, assume (iv) holds. Consider any $a, b \in B$. In case $(a/b) \in D^{\mathcal{B}}$, by (2.8) with $i = 0$ /"and (2.9)", we have $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.10), we get $D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas (5.7) is true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, while $\sim^{\mathfrak{B}} \sim^{\mathfrak{B}} c = c$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, we conclude that $(c \veebar^{\mathfrak{B}} d) = (\sim^{\mathfrak{B}} \sim^{\mathfrak{B}} c \veebar^{\mathfrak{B}} d) \notin D^{\mathcal{B}}$, for, otherwise, by (5.7), we would get $d \in D^{\mathcal{B}}$. Hence, by (2.9), we eventually get $(a \veebar^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$.] Thus, (ii) holds, as required. □

Lemma 5.27. *Let* $\mathcal{B}$ *be a* Σ*-matrix and* C' *the logic of* $\mathcal{B}$*. Suppose [either]* $\mathcal{B}$ *is false-singular (in particular,* $\sim$*-classical) [or both* $\mathcal{B}$ *is* $\sim$*-super-classical and* $|B| \leqslant 3$*]. Then, the following [but (i)] are equivalent:*

(i) C' *is weakly* $\sqsupset$*-implicative;*
(ii) C' *is* $\sqsupset$*-implicative;*
(iii) $\mathcal{B}$ *is* $\sqsupset$*-implicative;*
(iv) (2.12), (2.13) *and* (2.11) *[as well as both* (2.15) *and the* Ex Contradictione Quodlibet *axiom:*

$$\sim x_0 \sqsupset (x_0 \sqsupset x_1)] \tag{5.8}$$

are satisfied in C' *(viz., true in* $\mathcal{B}$*).*

In particular, any $\sim$*-classical/"three-valued* $\sim$*-paraconsistent"* Σ*-logic /"with subclassical negation* $\sim$*" is* $\sqsupset$*-implicative iff it is weakly so.*

Proof. First, (iii)⇒(ii) is immediate, while (i) is a particular case of (ii).

Next, assume (i[i]) holds. Then, (2.12), (2.13) and (2.11) [as well as (2.15)] are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, non-$\sim$-paraconsistent, and so is C'. Hence, by Deduction Theorem, (5.8) is satisfied in C'.] Thus, (iv) holds.

Finally, assume (iv) holds. Consider any $a, b \in B$. In case $b \in D^{\mathcal{B}}$, by (2.13) and (2.11), we have $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Likewise, in case $\{a, a \sqsupset^{\mathfrak{B}} b\} \subseteq D^{\mathcal{B}}$, by (2.11), we have $b \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.12), we get $D^{\mathcal{B}} \ni (a \sqsupset^{\mathfrak{B}} a) = (a \sqsupset^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas both (2.15) and (5.8) and true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, by (2.11) and (5.8), we conclude that $(c \sqsupset^{\mathfrak{B}} d) \in D^{\mathcal{B}}$. Let us prove, by contradiction, that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}}$. For suppose $(d \sqsupset^{\mathfrak{B}} c) \notin D^{\mathcal{B}}$, in which case $(d \sqsupset^{\mathfrak{B}} c) = (c/d)$, and so we have

$((d \sqsupset^{\mathfrak{B}} c) \sqsupset^{\mathfrak{B}} d) = ((c \sqsupset^{\mathfrak{B}} d)/(d \sqsupset^{\mathfrak{B}} d)) \in D^{\mathcal{B}}/$, by (2.12). Hence, by (2.11) and (2.15), we get $d \in D^{\mathcal{B}}$. This contradiction shows that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}} \ni (c \sqsupset^{\mathfrak{B}} d)$. In particular, we eventually get $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$.] Thus, (iii) holds, as required/, in view of Theorem 5.23. □

Three-valued logics with subclassical negation $\sim$ (even both implicative — in particular, disjunctive — and conjunctive ones) need not, generally speaking, be non-$\sim$-classical, as it ensues from the following elementary example:

Example 5.28. Let $\Sigma \triangleq \Sigma_{+,\sim}$ and $(\mathcal{B}/\mathcal{E})|\mathcal{F}$ the $\wedge$-conjunctive $\vee$-disjunctive $\sim$-negative "false-/truth-singular $\sim$-super-classical"|$\sim$-classical Σ-matrix with $(((\mathfrak{B}/\mathfrak{E})|\mathfrak{F})\restriction\Sigma_+) \triangleq \mathfrak{D}_{3|2}$. Then, $(\mathcal{B}/\mathcal{E})|\mathcal{F}$ is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq (\sim x_0 \vee x_1)$. And what is more, $\chi^{\mathcal{B}/\mathcal{E}} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}/\mathcal{E}, \mathcal{F})$. Therefore, by (2.20), $\mathcal{B}/\mathcal{E}$ define the same $\sim$-classical Σ-logic of $\mathcal{F}$. On the other hand, $\mathcal{B}$, being false-singular, is not isomorphic to $\mathcal{E}$, not being so. Moreover, $h \triangleq (\Delta_2 \circ \chi^{\mathcal{B}/\mathcal{E}})$ is a non-diagonal (for $h(\frac{1}{2}) = (1/0) \neq \frac{1}{2}$) strict homomorphism from $\mathcal{B}/\mathcal{E}$ to itself, so the "[]"-non-optional inclusion in (5.5) may be proper. □

On the other hand, $\sim$-classical three-valued Σ-logics with subclassical negation $\sim$ are self-extensional, in view of Example 4.2. This makes the following characterization especially acute:

Theorem 5.29. *Suppose $\mathcal{A}$ (viz., C; cf. Lemma 5.26) is $\veebar$-disjunctive. Then, the following are equivalent:*

(i) *C is $\sim$-classical;*
(ii) *$\mathcal{A}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix;*
(iii) *$\mathcal{A}$ is not simple;*
(iv) *$\mathcal{A}$ is not hereditarily simple;*
(v) *$\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$.*

Proof. First, (ii)⇒(i) is by (2.20). Conversely, assume (i) holds. Let $\mathcal{B}$ be any $\sim$-classical Σ-matrix defining C. Then, by Lemma 5.26, $\mathcal{B}$ is $\veebar$-disjunctive. Moreover, it is simple and has no proper submatrix. In this way, as $\mathcal{A}$ is a $\veebar$-disjunctive model of C, Lemmas 2.5, 3.13 and Remark 2.3(ii) yield (ii).

Next, (ii)⇒(iii) is by Remark 2.3(ii) and the fact that $3 \nleq 2$. Further, (iv) is a particular case of (iii). The converse is by the fact that any proper submatrix of $\mathcal{A}$, being either one-valued or $\sim$-classical, is simple. Furthermore, (iii)⇒(v) is by:

Claim 5.30. *Let $\mathcal{B}$ be a three-valued as well as both consistent and truth-non-empty Σ-matrix. Then, any non-diagonal congruence θ of it is equal to $\theta^{\mathcal{B}}$.*

Proof. First, we have $\theta \subseteq \theta^{\mathcal{B}}$. Conversely, consider any $\bar{a} \in \theta^{\mathcal{B}}$. Then, in case $a_0 = a_1$, we have $\bar{a} \in \Delta_B \subseteq \theta$. Otherwise, take any $\bar{b} \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case $\bar{b} \in \theta^{\mathcal{B}}$, for $\theta \subseteq \theta^{\mathcal{B}}$. Then, as $|B| = 3 \ngeq 4$, there are some $i, j \in 2$ such that $a_i = b_j$. Hence, if a_{1-i} was not equal to b_{1-j}, then we would have both $|\{a_i, a_{1-i}, b_{1-j}\}| = 3 = |B|$, in which case we would get $\{a_i, a_{1-i}, b_{1-j}\} = B$, and $\chi^{\mathcal{B}}(b_{1-j}) = \chi^{\mathcal{B}}(b_j) = \chi^{\mathcal{B}}(a_i) = \chi^{\mathcal{B}}(a_{1-i})$, and so $\mathcal{B}$ would be either truth-empty or inconsistent. Therefore, both $a_{1-i} = b_{1-j}$ and $a_i = b_j$. Thus, since θ is symmetric, we eventually get $\bar{a} \in \theta$, for $\bar{b} \in \theta$, as required. □

Finally, assume (v) holds. Then, $\theta \triangleq \theta^{\mathcal{A}}$, including itself, is a congruence of $\mathcal{A}$, in which case $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}/\theta)$, while $\mathcal{A}/\theta$ is $\sim$-classical, and so (ii) holds. □

Corollary 5.31. *Let $\mathcal{B}$ be a $\sim$-super-classical Σ-matrix. Suppose C is defined by $\mathcal{B}$ as well as both $\veebar$-disjunctive and non-$\sim$-classical. Then, $\mathcal{B}$ is isomorphic to $\mathcal{A}$.*

Proof. In that case, both $\mathcal{A}$ and $\mathcal{B}$ are both $\veebar$-disjunctive and hereditarily simple, by Lemma 5.26 and Theorem 5.29. Therefore, by Lemmas 2.5, 3.13 and Remark 2.3(ii), $\mathcal{A}$ is isomorphic to a submatrix of $\mathcal{B}$, and so to $\mathcal{B}$ itself, for $|A| = 3 = |B|$ is finite. □

In view of Theorem 5.23 [and Corollary 5.31 as well as Lemma 5.26], any [non-$\sim$-classical $\veebar$-disjunctive] three-valued Σ-logic with subclassical negation $\sim$ is defined by a [unique (either up to isomorphism or when dealing with merely *canonical* three-valued $\sim$-super-classical Σ-matrices, i.e., those of the form $\mathcal{A}'$ with $A' = (3 \div 2)$ and $a_{\mathcal{A}'} = a$, for all $a \in A'$, in which case isomorphic ones are equal, by (5.5) applied to their common $\sim$-reduct) $\veebar$-disjunctive] three-valued $\sim$-super-classical Σ-matrix [the unique canonical one being said to be *characteristic for|of* the logic]. On the other hand, such is not the case for $\sim$-classical (even both implicative — in particular, disjunctive — and conjunctive) ones, in view of Theorem 5.23 and Example 5.28.

Corollary 5.32. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Suppose $\mathcal{A}$ is $\veebar$-disjunctive (viz., C is so; cf. Lemma 5.26) and simple (i.e., C is not $\sim$-classical; cf. Theorem 5.29) [as well as canonical]. Then, C' is defined by a [unique] Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, $\sim$ is a subclassical negation for C'. Hence, by Theorem 5.23, C' is defined by a $\sim$-super-classical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the $\sim$-super-classical Σ-matrix $\mathcal{A}' \restriction \Sigma$, and so, by Theorem 5.31, there is some isomorphism e from $(\mathcal{A}' \restriction \Sigma)$ onto $\mathcal{A}$, in which case it is an isomorphism from $\mathcal{A}'$ onto the Σ'-expansion $\mathcal{A}'' \triangleq \langle e[\mathfrak{A}'], e[D^{\mathcal{A}'}]\rangle$ of $\mathcal{A}$, and so, by (2.20), C' is defined by $\mathcal{A}''$. [Finally, (5.5) and Corollary 5.31 complete the argument.] □

And what is more, taking Lemma 5.5 into account, it is worth to explore connections between self-extensionality and existence of a classical extension. This makes the following characterization of the latter especially acute:

Theorem 5.33. *Let $\mathcal{B}$ be a $\sim$-classical extension of C. Suppose C is $\veebar$-disjunctive and not $\sim$-classical. Then, 2 forms a subalgebra of $\mathcal{A}$, $\mathcal{A} \restriction 2$ being isomorphic to $\mathcal{B}$. In particular, C, being $\veebar$-disjunctive, is $\sim$-subclassical iff either of the following holds:*

(i) *C is $\sim$-classical;*
(ii) *2 forms a subalgebra of $\mathcal{A}$, in which case $\mathcal{A} \restriction 2$ is a $\sim$-classical model of C isomorphic to any other one, and so defines a unique $\sim$-classical extension of C.*

Proof. In that case, by Lemma 5.26 and Theorem 5.29, $\mathcal{A}$ is $\veebar$-disjunctive and hereditarily simple. Likewise, $\mathcal{B}$ is simple and, by Lemma 5.26, $\veebar$-disjunctive. Hence, by Lemmas 2.5, 3.13 and Remark 2.3(ii), there is some embedding e of $\mathcal{B}$ into $\mathcal{A}$, in which case $\operatorname{img} e$ forms a two-element subalgebra of $\mathfrak{A}$, and so $(\operatorname{img} e) = 2$, e being an isomorphism from $\mathcal{B}$ onto $\mathcal{A} \restriction 2$. In this way, (2.20) completes the argument. □

Next, we have the *dual* three-valued $\sim$-super-classical Σ-matrix $\partial(\mathcal{A}) \triangleq \langle \mathfrak{A}, \{1\} \cup (\{\frac{1}{2}\} \cap (A \setminus D^{\mathcal{A}}))\rangle$, in which case it is false/truth-singular iff $\mathcal{A}$ is not so, while:

$$(\theta^{\mathcal{A}} \cap \theta^{\partial(\mathcal{A})}) = \Delta_A. \tag{5.9}$$

Likewise, set $\mathcal{A}_{a[+(b)]} \triangleq \langle \mathfrak{A}, \{[\frac{1}{2}(-\frac{1}{2} + b),]a\}\rangle$, where $a[(, b)] \in A$, in which case $(\partial(\mathcal{A})/\mathcal{A}) = \mathcal{A}_{1[+]}$, whenever $\mathcal{A}$ is [not] false-/truth-singular, while:

$$(\theta^{\mathcal{A}_{a[+]}} \cap \theta^{\mathcal{A}_{b[+]}}) = \Delta_A, \tag{5.10}$$

for all distinct $a, b \in A$.

Further, given any $i \in 2$, put $h_i \triangleq (\Delta_2 \cup \{\langle \frac{1}{2}, i \rangle\}) : (3 \div 2) \to 2$, in which case:

$$h_{0/1}^{-1}[D^{\mathcal{A}}] = D^{\partial(\mathcal{A})}, \tag{5.11}$$

whenever $\mathcal{A}$ is false-/truth-singular.

Finally, let $h_{1-} : (3 \div 2) \to (3 \div 2), a \mapsto (1 - a)$, in which case:

$$h_{1-}^{-1}[D^{\mathcal{A}_{i[+]}}] = D^{\mathcal{A}_{(1-i)[+]}}, \tag{5.12}$$

for all $i \in 2$.

5.2.1. *Both conjunctive and disjunctive logics.*

Lemma 5.34. *Suppose C is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 5.26) as well as both self-extensional and not $\sim$-classical. Then, $\mathfrak{A}$ is a distributive $(\bar{\wedge}, \veebar)$-lattice with zero 0 and unit 1.*

Proof. In that case, $\mathcal{A}$ is both false-/truth-singular, in which case it (and so C) is not $(\veebar, \sim)$-paracomplete/$\sim$-paraconsistent, and, by Theorem 5.29, simple, in which case, as A is finite, by Theorem 4.5(i)⇒(iv), $\mathfrak{A}$ is a distributive $(\bar{\wedge}, \veebar)$-lattice with zero and unit, and so, by the following claim, $\flat^{\mathfrak{A}}_{\bar{\wedge}|\veebar} \notin | \in D^{\mathcal{A}}$:

Claim 5.35. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical $\bar{\wedge}$-conjunctive $\veebar$-disjunctive Σ-matrix. Suppose $\mathfrak{B}$ is a $(\bar{\wedge}, \veebar)$-lattice. Then, $\flat^{\mathfrak{B}}_{\bar{\wedge}|\veebar} \notin | \in D^{\mathcal{B}}$. In particular, $\flat^{\mathfrak{B}}_{\bar{\wedge}/\veebar} = (0/1)$, whenever $\mathcal{B}$ is false-/truth-singular.*

Proof. In that case, by the $\bar{\wedge}$-conjunctivity|$\veebar$-disjunctivity of $\mathcal{B}$, since $(0|1) \notin | \in D^{\mathcal{B}}$, we have $\flat^{\mathfrak{B}}_{\bar{\wedge}|\veebar} = ((0|1)(\bar{\wedge}|\veebar)^{\mathfrak{B}}\flat^{\mathfrak{B}}_{\bar{\wedge}|\veebar}) \notin | \in D^{\mathcal{B}}$, as required. □

In particular, $\flat^{\mathfrak{A}}_{\bar{\wedge}/\veebar} = (0/1)$, in which case $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, and so, if $\flat^{\mathfrak{A}}_{\veebar/\bar{\wedge}}$ was equal to $\frac{1}{2}$, then $\mathcal{A}_{\frac{1}{2}/(0+1)}$ would be both $\bar{\wedge}$-conjunctive and truth-non-empty as well as $(\veebar, \sim)$-paracomplete/$\sim$-paraconsistent, for (2.17)/(2.16) would not be true in it under $[x_0/0]/[x_0/0, x_1/\frac{1}{2}]$, respectively, in which case, by Theorem 4.5(i)⇒(iv), it would be a model of C, and so this would be $(\veebar, \sim)$-paracomplete/$\sim$-paraconsistent. Thus, $\flat^{\mathfrak{A}}_{\veebar/\bar{\wedge}} = (1/0)$, as required. □

As for negative instances of Lemma 5.34, as a first one, we should like to highlight P^1 [22] (cf. [13]), in which case $\mathfrak{A}$ has no semi-lattice (even merely idempotent and commutative) secondary operations, simply because the values of primary ones belong to $2 \not\ni \frac{1}{2}$, in which case 2 forms a subalgebra of $\mathfrak{A}$, and so $\mathcal{A}$, being $\supset$-implicative, is both $\uplus_{\supset}$-disjunctive and $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \supset \sim(x_0 \supset x_0))$ (in particular, this is $\uplus^{\neg}_{\supset}$-conjunctive; cf. Remark 2.4(i)**a**)). Likewise, three-valued expansions of HZ [6] are not self-extensional, because, in that case, though $\mathcal{A}$, being false-singular, is neither $\wedge$-conjunctive nor $\vee$-disjunctive, simply because $\mathfrak{A}$ is a $(\wedge, \vee)$-lattice but with distinguished zero $\frac{1}{2}$, $\mathfrak{A}$ is a $(\vee^{\sim}, \wedge^{\sim})$-lattice with zero 0 and unit $\frac{1}{2} \neq 1$, in which case $\mathcal{A}$ is both $\vee^{\sim}$-conjunctive and $\wedge^{\sim}$-disjunctive. On the other hand, arbitrary three-valued expansions of both P^1 and HZ are covered by the next subsection as well, the latter ones being equally covered by the following characterization (more precisely, some of its consequences, as we show below):

Theorem 5.36. *Suppose C is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 5.26) as well as both C is not $\sim$-classical and $\mathcal{A}$ is false-/truth-singular. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$h_{0/1}$ is an endomorphism of $\mathfrak{A}$;*
(iii) *$\partial(\mathcal{A}) \in \mathrm{Mod}(C)$;*
(iv) *$\mathfrak{A}$ is a [distributive] $(\bar{\wedge}, \veebar)$-lattice {with zero 0 and unit 1} having a non-singular non-diagonal (partial) endomorphism.*

Proof. In that case, by Theorem 5.29, $\mathcal{A}$ is hereditarily simple.

First, assume (i) holds. Then, by Lemma 5.34, $\mathfrak{A}$ is a $(\overline{\wedge}, \underline{\vee})$-lattice with zero 0 and unit 1. Moreover, as $\frac{1}{2} \neq (1/0)$, by Theorem 4.6, there is some non-singular $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(\frac{1}{2})) \neq \chi^{\mathcal{A}}(h(1/0))$. Then, $B \triangleq (\operatorname{img} h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, in which case $2 \subseteq B$, and so $\mathfrak{B} \triangleq (\mathfrak{A} \upharpoonright B)$ is a $(\overline{\wedge}, \underline{\vee})$-lattice with zero|unit 0|1. Hence, since $h \in \hom(\mathfrak{A}, \mathfrak{B})$ is surjective, by Lemma 2.1, we conclude that $h \upharpoonright 2$ is diagonal, in which case $h(1/0) = (1/0) \in / \notin D^{\mathcal{A}}$, and so $h(\frac{1}{2}) \notin / \in D^{\mathcal{A}}$. In this way, $h(\frac{1}{2}) = (0/1)$, in which case $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{0/1}$, and so (ii) holds.

Next, (ii)⇒(iii) is by (2.20) and (5.11). Further, (iii)⇒(i) is by (5.9) and Theorem 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \partial(\mathcal{A})\}$. Thus, we have proved the equivalence of (i,ii,iii). Furthermore, (i,ii)⇒(iv) is by Lemma 5.34 and the fact that $h_{0/1}(\frac{1}{2}) \in 2 \not\ni \frac{1}{2}$, while $(\operatorname{img} h_{0/1}) = 2$ is not a singleton.

Finally, assume (iv) holds. Then, there are some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and some non-diagonal non-singular $h \in \hom(\mathfrak{B}, \mathfrak{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, and so does $B = (\operatorname{dom} h)$. Hence, $2 \subseteq (B \cap D)$, in which case, by Claim 5.35, both $\mathfrak{B}$ and $\mathfrak{D} \triangleq (\mathfrak{A} \upharpoonright D)$ are $(\overline{\wedge}, \underline{\vee})$-lattices with zero/unit 0/1, and so, as $h \in \hom(\mathfrak{B}, \mathfrak{D})$ is surjective, by Lemma 2.1, $h(0/1) = (0/1)$, in which case $(1/0) = \sim^{\mathfrak{A}}(0/1) = \sim^{\mathfrak{A}} h(0/1) = h(\sim^{\mathfrak{A}}(0/1)) = h(1/0)$, and so $h \upharpoonright 2$ is diagonal. Therefore, $B = A$, while $h(\frac{1}{2}) \neq \frac{1}{2}$. In this way, if $h(\frac{1}{2})$ was equal to 1/0, then h would be a non-injective strict homomorphism from $\mathcal{A}$ to itself, in which case, by Remark 2.3(ii), $\mathcal{A}$ would not be simple. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{0/1}$, so (ii) holds, as required. □

First, by Lemma 4.14 and Theorem 5.36(i)⇔(iv), we immediately have:

Corollary 5.37. *Suppose $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., C is so; cf. Lemma 5.26) as well as either $\sim$-paraconsistent or $(\underline{\vee}, \sim)$-paracomplete (in which case C is so, and so is not $\sim$-classical, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$). Then, C is self-extensional iff the following hold:*

(i) *$\mathcal{A}$ has no equational implication;*

(ii) *$\mathfrak{A}$ is a (distributive) $(\overline{\wedge}, \underline{\vee})$-lattice [with zero 0 and unit 1].*

In view of Theorems 10, 13 and Example 10 of [19], this positively covers [the implication-less fragment of] Gödel's three-valued logic [4] as well as their "$\sim$-paraconsistent counterparts" resulted from lattice duality — viz., using dual (relative) pseudo-complement(s) instead of the direct one(s). As to negative instances of Theorem 5.36, we need some its generic consequences.

As $(\operatorname{img} h_{0/1}) = 2$, by Theorems 5.33 and 5.36(i)⇒(ii), we first have:

Corollary 5.38. *Suppose C is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 5.26) as well as self-extensional. Then, C is $\sim$-subclassical.*

Next, we have:

Corollary 5.39. *Suppose $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., C is so; cf. Lemma 5.26) [as well as not $\sim$-negative (in particular, either $\sim$-paraconsistent or $(\underline{\vee}, \sim)$-paracomplete {viz., C is so}), unless C is $\sim$-classical]. Then, C is self-extensional [if and] only if the following hold:*

(i) *C has PWC with respect to $\sim$;*

(ii) *either C is $\sim$-classical or $\mathfrak{A}$ is a $(\overline{\wedge}, \underline{\vee})$-lattice.*

Proof. First, assume C is self-extensional. Consider the following complementary cases:

- C is $\sim$-classical,
  in which case (ii) holds, while, by Remark 2.4(i)**b)**, (i) holds too.

- C is not $\sim$-classical.
  Then, by Lemma 5.34, $\mathfrak{A}$ is a $(\overline{\wedge}, \underline{\vee})$-lattice with $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} \frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, in which case (ii) holds, while $\sim^{\mathfrak{A}}$ is anti-monotonic with respect to $\leq^{\mathfrak{A}}_{\overline{\wedge}}$, and so, by Theorem 4.5(i)⇒(ii), (i) holds too.

[Conversely, assume (i,ii) hold. Consider the following complementary cases:

- C is $\sim$-classical.
  Then, by Example 4.2, it is self-extensional.
- C is not $\sim$-classical.
  Then, by (ii), $\mathfrak{A}$ is a $(\overline{\wedge}, \underline{\vee})$-lattice, while $\mathcal{A}$ is non-$\sim$-negative as well as false-/truth-singular, in which case $\sim^{\mathfrak{A}} \frac{1}{2} \neq (0/1)$, and so $D^{\partial(\mathcal{A})} = (\sim^{\mathfrak{A}})^{-1}[A \setminus D^{\mathcal{A}}]$. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C(\phi)$, in which case $\sim\phi \in C(\sim\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h(\phi) \in D^{\partial(\mathcal{A})}$, in which case $h(\sim\phi) \notin D^{\mathcal{A}}$, and so $h(\sim\psi) \notin D^{\mathcal{A}}$, that is, $h(\psi) \in D^{\partial(\mathcal{A})}$. Thus, $\partial(\mathcal{A})$ is a $(2 \setminus 1)$-model of C. In particular, for each $i \in 2$, the unary Σ-rule $(x_0 \mathbin{\overline{\wedge}} x_1) \vdash x_i$, being satisfied in C, for this is $\overline{\wedge}$-conjunctive, is true in $\partial(\mathcal{A})$. Conversely, consider any $\bar{a} \in (D^{\partial(\mathcal{D})})^2$. Then, in case $a_0 = a_1$, by the idempotencity identity for $\overline{\wedge}$, we have $(a_0 \mathbin{\overline{\wedge}^{\mathfrak{A}}} a_1) = a_0 \in D^{\partial(\mathcal{A})}$. Otherwise, $D^{\partial(\mathcal{A})} = (\mathrm{img}\, \bar{a})$ is not a singleton, in which case we have $D^{[\partial](\mathcal{A})} = \{1[, \frac{1}{2}]\}$, and so, by the $\underline{\vee}$-disjunctivity of $\mathcal{A}$, we get $(a_0 \mathbin{\underline{\vee}^{\mathfrak{A}}} a_1) = 1$, that is, $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, in which case we eventually get $(a_0 \mathbin{\overline{\wedge}^{\mathfrak{A}}} a_1) = \frac{1}{2} \in D^{\partial(\mathcal{A})}$ too. Thus, $\partial(\mathcal{A})$ is $\overline{\wedge}$-conjunctive, in which case, by Lemma 4.3, it, being truth-non-empty, is a model of C, and so, by Theorem 5.36(iii)⇒(i), C is self-extensional.] □

Corollary 5.40. *Suppose C is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 5.26) as well as self-extensional. Then, $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$.*

Proof. Consider the following complementary cases:

- C is $\sim$-classical.
  Then, by Remark 2.4(ii) and Theorem 5.29, $\mathcal{A}$ is $\sim$-negative, in which case $(\sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}}) \Leftrightarrow (\frac{1}{2} \notin D^{\mathcal{A}})$, and so $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$.
- C is not $\sim$-classical.
  Then, by Theorem 5.36(i)⇒(ii), $h \triangleq h_i \in \hom(\mathfrak{A}, \mathfrak{A})$, for some $i \in 2$, in which case $h(\frac{1}{2}) = i$, and so, if $\sim^{\mathfrak{A}} \frac{1}{2}$ was equal to $\frac{1}{2}$, then we would have $(1 - i) = \sim^{\mathfrak{A}} i = \sim^{\mathfrak{A}} h(\frac{1}{2}) = h(\sim^{\mathfrak{A}} \frac{1}{2}) = h(\frac{1}{2}) = i$. Thus, $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$. □

This negatively covers arbitrary three-valued expansions (cf. Corollary 5.32 in this connection) of both Kleene's three-valued logic [7] (including those of Łukasiewicz' one L_3 [9]) and LP [12] (including those of the *logic of antinomies* LA [1]) as well as of HZ. On the other hand, three-valued expansions of Ł$_3$, LA and HZ are equally covered by the next subsubsection.

5.2.2. *Implicative logics.*

Lemma 5.41. *Suppose C is $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 5.27) and not $\sim$-classical. Then, $h_i \in \hom(\mathfrak{A}, \mathfrak{A})$, for no $i \in 2$.*

Proof. By contradiction. For suppose $h_i \in \hom(\mathfrak{A}, \mathfrak{A})$, for some $i \in 2$, in which case $(\ker h_i) \in \mathrm{Con}(\mathfrak{A})$, and so, if i was equal to $1/0$, whenever $\mathcal{A}$ was false-/truth-singular, then $\theta^{\mathcal{A}}$ would be equal to $(\ker h_i) \in \mathrm{Con}(\mathfrak{A})$, contrary to Theorem 5.29, while $2 = (\mathrm{img}\, h_i)$ forms a subalgebra of $\mathfrak{A}$, and so $((0/1) \sqsupset^{\mathfrak{A}} 0) = (1/0)$, whenever $\mathcal{A}$ is false-/truth-singular. Therefore, $i = (0/1)$, whenever $\mathcal{A}$ is false-/truth-singular, in which case $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = (0/1)$, and so $(0/1) = h_i(0/1) = h_i(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = (h_i(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h_i(0)) = ((0/1) \sqsupset^{\mathfrak{A}} 0) = (1/0)$. This contradiction completes the argument. □

By Theorem 5.36(i)⇒(ii) and Lemma 5.41, we immediately have:

Corollary 5.42. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative (and so $\uplus_{\sqsupset}$-disjunctive) and conjunctive (in particular, negative; cf. Remark 2.4(i)***a***)). Then, C is not self-extensional, unless it is $\sim$-classical.*

This immediately both shows that Gödel's three-valued logic [4], though being weakly implicative, is not implicative, and covers three-valued expansions of L_3, LA, HZ and P^1, those of the former being equally covered by:

Corollary 5.43. *Suppose $\mathcal{A}$ is both truth-singular (in particular, both $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete) and $\sqsupset$-implicative. Then, C is not self-extensional, unless it is $\sim$-classical.*

Proof. Then, $(a \sqsupset^{\mathfrak{A}} a) = 1$, for all $a \in A$, in which case $\mathcal{A}$ is $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, and so Corollary 5.42 completes the argument. □

The "false-singular" case is but more complicated. First, we have:

Corollary 5.44. *Suppose $\mathcal{A}$ is both false-singular and $\sqsupset$-implicative. Then, C is not self-extensional, unless it is either $\sim$-paraconsistent or $\sim$-classical.*

Proof. If C is not $\sim$-paraconsistent, then $\sim^{\mathfrak{A}}\frac{1}{2} = 0$, in which case $\mathcal{A}$ is $\sim$-negative, and so Corollary 5.42 completes the argument. □

Theorem 5.45. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative (viz., C is so; cf. Lemma 5.27), hereditarily simple (i.e., C is not $\sim$-classical; cf. Theorem 5.29) and false-singular (in particular, $\sim$-paraconsistent [i.e., C is so]). Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$;*
(iii) *$\sim^{\mathfrak{A}}$ is a bijective endomorphism of $\mathfrak{A}$;*
(iv) *h_{1-} is an endomorphism of $\mathfrak{A}$;*
(v) *$\mathcal{A}_{+0}$ is isomorphic to $\mathcal{A}$;*
(vi) *C is defined by $\mathcal{A}_{0+}$;*
(vii) *$\mathcal{A}_{0+} \in \mathrm{Mod}(C)$;*
(viii) *$\mathfrak{A}$ is an $\sqsupset$-implicative inner semilattice having a non-singular non-diagonal (partial) endomorphism.*

Proof. First, assume (i) holds. Then, as $\frac{1}{2} \neq 1$, by Theorem 4.10, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(\frac{1}{2})) \neq \chi^{\mathcal{A}}(h(1))$. Moreover, by (2.12), $a \triangleq (\frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. If a was not equal to $\frac{1}{2}$, then it would be equal to 1, and so would be $(b \sqsupset^{\mathfrak{A}} b)$, for any $b \in A$, in view of (2.12) and Lemma 4.7, in which case $\mathcal{A}$ would be $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, contrary to Corollary 5.42. Therefore, $a = \frac{1}{2}$, in which case $(b \sqsupset^{\mathfrak{A}} b) = \frac{1}{2}$, for any $b \in A$, in view of (2.12) and Lemma 4.7, and so $h(\frac{1}{2}) = (h(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h(\frac{1}{2})) = \frac{1}{2} \in D^{\mathcal{A}}$. Hence, $h(1) \notin D^{\mathcal{A}}$, in which case $h(1) = 0$, and so $h(0) = h(\sim^{\mathfrak{A}}1) = \sim^{\mathfrak{A}}h(1) = \sim^{\mathfrak{A}}0 = 1$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{1-}$, and so (iv) holds.

Next, (iv)⇒(v/iii) is by the fact that $h_{1-} : A \to A$ is bijective and (5.12)/:

Claim 5.46. *Suppose $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$. Then, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.*

Proof. If $\sim^{\mathfrak{A}}\frac{1}{2}$ was not equal to $\frac{1}{2}$, then it would be equal to some $i \in 2$, in which case we would have $(1 - i) = h_{1-}(i) = h_{1-}(\sim^{\mathfrak{A}}\frac{1}{2}) = \sim^{\mathfrak{A}}h_{1-}(\frac{1}{2}) = \sim^{\mathfrak{A}}\frac{1}{2} = i$. □

Conversely, assume (iii) holds. Then, $\sim^{\mathfrak{A}}[A/2] = (A/2)$, in which case $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, and so $h_{1-} = \sim^{\mathfrak{A}} \in \hom(\mathfrak{A}, \mathfrak{A})$. Thus, (iv) holds.

Further, (v)⇒(vi) is by (2.20), while (vii) is a particular case of (vi), whereas (vii)⇒(ii) is by the fact that $D^{\mathcal{A}_{\frac{1}{2}}} = (D^{\mathcal{A}} \cap D^{\mathcal{A}_{0+}})$. Furthermore, (ii)⇒(i) is by

(5.10) and Theorem 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \mathcal{A}_{\frac{1}{2}}\}$. Thus, we have proved the equivalence of (i-vii).

Finally, (i,iv)⇒(viii) is by Theorem 4.9 and the fact that $h_{1-}(0) = 1 \neq 0$, while $(\mathrm{img}\, h_{1-}) = A$ is not a singleton. Conversely, assume (viii) holds. Then, $\mathfrak{A}$ is an $\sqsupset$-implicative inner semi-lattice, while there are some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and some non-singular non-diagonal $h \in \hom(\mathfrak{B}, \mathfrak{A})$, in which case $(\mathrm{img}\, h) \neq \varnothing$ is not a singleton, and so is $B = (\mathrm{dom}\, h) \neq \varnothing$. Hence, $2 \subseteq B$, in which case, $a \triangleq (1 \sqsupset^{\mathfrak{A}} 1) \in B$, and so, by (2.3), $h(a) = (h(1) \sqsupset^{\mathfrak{A}} h(1)) = a$. Moreover, by (2.12), $a \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. Therefore, if a was not equal to $\frac{1}{2}$, then it would be equal to 1, in which case we would have $h(1) = 1$, and so would get $h(0) = h(\sim^{\mathfrak{A}} 1) = \sim^{\mathfrak{A}} h(1) = \sim^{\mathfrak{A}} 1 = 0$, in which case, by the non-diagonality of h, we would have $\frac{1}{2} \in B$ and $h(\frac{1}{2}) = i$, for some $i \in 2$, and so $h = h_i$ would be an endomorphism of $\mathfrak{A}$, contrary to Lemma 5.41. Thus, $B \ni a = \frac{1}{2}$, in which case $B = A$, while $h(\frac{1}{2}) = \frac{1}{2}$, and so, by the non-diagonality of h, there is some $i \in 2$ such that $h(i) \neq i$. Let us prove, by contradiction, that $h(i) \neq \frac{1}{2}$. For suppose $h(i) = \frac{1}{2}$. In that case, if $h(1-i)$ was not equal to $\frac{1}{2}$, then it would be equal to some $j \in 2$, and so we would have $\frac{1}{2} = h(i) = h(\sim^{\mathfrak{A}}(1-i)) = \sim^{\mathfrak{A}} h(1-i) = \sim^{\mathfrak{A}} j = (1-j) \in 2$. Hence, $h(1-i) = \frac{1}{2}$, in which case, as $(\mathrm{dom}\, h) = B = A$ and $\{i, 1-i\} = 2 = (A \setminus \{\frac{1}{2}\})$, we get $(\mathrm{img}\, h) = \{\frac{1}{2}\}$, contrary to the non-singularity of h. Thus, $h(i) \neq \frac{1}{2}$, in which case $h(i) = (1-i)$, and so $h(1-i) = h(\sim^{\mathfrak{A}} i) = \sim^{\mathfrak{A}} h(i) = \sim^{\mathfrak{A}}(1-i) = i$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{1-}$, and so (iv) holds, as required. □

First, by Remark 4.13(v),(iii)**a)**, Lemma 4.14, Corollary 5.43 and Theorem 5.45(i)⇔(viii), we immediately have:

Corollary 5.47. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative (viz., C is so; cf. Lemma 5.27) as well as either $\sim$-paraconsistent or both $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (in which case C is so [cf. Lemma 5.26], and so is not $\sim$-classical, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$). Then, C is self-extensional iff the following hold:*

(i) *$\mathcal{A}$ has no equational implication;*
(ii) *$\mathfrak{A}$ is an $\sqsupset$-implicative inner semi-lattice.*

Next, as opposed to Corollary 5.38, we have:

Corollary 5.48. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 5.27) and self-extensional. Then, it is $\sim$-subclassical iff it is $\sim$-classical.*

Proof. The "if" part is immediate. Conversely, if C was $\sim$-subclassical but not $\sim$-classical, then, by Theorem 5.33, 2 would form a subalgebra of $\mathfrak{A}$, while, by Corollary 5.43 and Theorem 5.45(i)⇒(ii), $\mathcal{A}_{\frac{1}{2}}$ would be a model of C, in which case, by (2.20), $\mathcal{A}_{\frac{1}{2}} \upharpoonright 2$ would be a truth-empty model of C, and so this would be theorem-less, contrary to (2.12)'s being a theorem of C. □

Likewise, as opposed to Corollary 5.40, by Corollary 5.43, Theorem 5.45(i)⇒(iv) and Claim 5.46, we have:

Corollary 5.49. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 5.27), non-$\sim$-classical and self-extensional. Then, $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.*

Furthermore, as opposed to Corollary 5.39, we have:

Corollary 5.50. *Suppose $\sqsupset \in \Sigma$ and C is $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 5.27). Then, C has PWC with respect to $\sim$ iff $\mathcal{A}$ is $\sim$-negative. In particular, any three-valued implicative $\sim$-paraconsistent/"both $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete" Σ-logic with subclassical negation $\sim$ does not have PWC w.r.t. $\sim$.*

Proof. The "if" part is by Remark 2.4(i)**b)**. The converse is proved by contradiction. For suppose C has PWC w.r.t. $\sim$ but $\mathcal{A}$ is not $\sim$-negative. Let $\Sigma' \triangleq \{\sqsupset, \sim\} \subseteq \Sigma$, in which case $\mathcal{A}' \triangleq (\mathcal{A}{\upharpoonright}\Sigma')$ is both three-valued, $\sim$-super-classical, $\sqsupset$-implicative and non-$\sim$-negative as well as defines the Σ'-fragment C' of C, and so C' is both $\sqsupset$-implicative and, by Remark 2.4(ii) and Theorem 5.29(i)$\Rightarrow$(ii), non-$\sim$-classical, for $\mathcal{A}'$ is non-$\sim$-negative, as well as has PWC w.r.t. $\sim$. In particular, for any $\langle\phi, \psi\rangle \in \equiv^{\omega}_{C'}$ and any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, we have both $\sim\phi \equiv^{\omega}_{C'} \sim\psi$, $(\phi \sqsupset \varphi) \equiv^{\omega}_{C'} (\psi \sqsupset \varphi)$ and $(\varphi \sqsupset \phi) \equiv^{\omega}_{C'} (\varphi \sqsupset \psi)$. Therefore, C' is self-extensional. Hence, as (2.12) is a theorem of C', by Corollary 5.43 and Theorem 5.45(i)$\Rightarrow$(ii), for every $a \in A$, $(a \sqsupset^{\mathfrak{A}} a) = \frac{1}{2}$, in which case, by Corollary 5.44, $\sim^{\mathfrak{A}}(a \sqsupset^{\mathfrak{A}} a) = \sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}}$, and so both $x_0 \sqsupset x_0$ and $\sim(x_0 \sqsupset x_0)$ are theorems of C'. Then, we have $(x_0 \sqsupset x_0) \in C'(\varnothing) \subseteq C'(x_0)$, in which case, by PWC w.r.t. $\sim$, we get $\sim x_0 \in C'(\sim(x_0 \sqsupset x_0)) \subseteq C'(\varnothing) \subseteq C'(x_0)$, and so, by (2.21) with $n = 0$ and $m = 1$, $\sim$ is not a subclassical negation for C'. In this way, Theorem 5.23 /"and Lemma 5.26" does/do complete the argument. □

Finally, existence of a self-extensional $\sqsupset$-implicative $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is due to Theorem 5.23 and:

Example 5.51. Let $\mathcal{A}$ be both canonical and false-singular, $\Sigma \triangleq \{\supset, \sim\}$ with binary $\supset$, $\sim^{\mathfrak{A}} \triangleq h_{1-}$ and $\supset^{\mathfrak{A}} \triangleq ((\Delta_A \times \{\frac{1}{2}\}) \cup (\pi_1{\upharpoonright}(A^2 \setminus \Delta_A)))$. Then, $\mathcal{A}$ is both $\supset$-implicative and $\sim$-paraconsistent, and so is C. And what is more, $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$, and so, by Theorem 5.45(iv)$\Rightarrow$(i), C is self-extensional. On the other hand, let $\mathcal{B}$ be any more $\sim$-super-classical $\supset$-implicative canonical Σ-matrix, the logic of which is self-extensional and not $\sim$-classical, in which case, by Corollary 5.43, $\mathcal{B}$ is false-singular, while, by Corollary 5.49, $\sim^{\mathfrak{B}}\frac{1}{2} = \frac{1}{2}$, and so $\sim^{\mathfrak{B}} = \sim^{\mathfrak{A}}$. Then, by Theorem 5.45(i)$\Rightarrow$(ii,iv,viii) and (2.12), $\mathfrak{B}$ is an $\sqsupset$-implicative semi-lattice, being a $\uplus_{\supset}$-semilattice with zero $\frac{1}{2} = (a \supset^{\mathfrak{B}} a)$, for all $a \in A$, and endomorphism h_{1-}. In particular, by (2.4), $(\frac{1}{2} \supset^{\mathfrak{B}} i) = i$, for all $i \in 2$. Moreover, by the $\supset$-implicativity of $\mathcal{B}$, we have $(1 \supset^{\mathfrak{B}} 0) = 0$, in which case $1 = h_{1-}(0) = h_{1-}(1 \supset^{\mathfrak{B}} 0) = (h_{1-}(1) \supset^{\mathfrak{B}} h_{1-}(0)) = (0 \supset^{\mathfrak{B}} 1)$, and $b \triangleq (1 \supset^{\mathfrak{B}} \frac{1}{2}) \in D^{\mathcal{B}} = \{\frac{1}{2}, 1\}$, in which case, if b was not equal to $\frac{1}{2}$, then it would be equal to 1, in which case we would have $\frac{1}{2} = (1 \uplus^{\mathfrak{B}}_{\supset} \frac{1}{2}) = (b \supset^{\mathfrak{B}} \frac{1}{2}) = b = 1$, and so $b = \frac{1}{2}$. Hence, $\frac{1}{2} = h_{1-}(\frac{1}{2}) = h_{1-}(b) = (h_{1-}(1) \supset^{\mathfrak{B}} h_{1-}(\frac{1}{2})) = (0 \supset^{\mathfrak{B}} \frac{1}{2})$. Thus, $(c \supset^{\mathfrak{B}} d) = (\frac{1}{2}/d)$, for all $c, d \in B$ such that $c = / \neq d$, in which case $\supset^{\mathfrak{B}} = \supset^{\mathfrak{A}}$, and so $\mathcal{B} = \mathcal{A}$. In this way, by (2.20), Theorem 5.23 and Lemma 5.27, the above C is a unique three-valued $\supset$-implicative self-extensional non-$\sim$-classical (in particular, $\sim$-paraconsistent) Σ-logic with subclassical negation $\sim$. □

This definitely shows that the justice is, at least, in that, when crooks (like Avron and Beziau et al.) plagiarize somebody else's labor (mine, in that case) and rewrite the genuine history of science for their exclusive benefit (in particular, by means of publishing plagiarized work backdating), they inevitably lose the capability (if any was at all ever) of obtaining and publishing new *and correct* results.

6. Conclusions

Aside from quite useful general results and their equally illustrative generic applications (sometimes, even multiple ones providing different insights) to infinite classes of particular logics, the paper demonstrates the value of the conception of equality determinant going back to [18, 19].

Among other things, deep connections between the self-extensionality of unitary finitely-valued logics with unary unitary equality determinant as well as "lattice conjunction and disjunction"/"implicative inner semi-lattice implication" and the algebraizability (in the sense of [16]) of two-side sequent calculi (associated according to [18]) and equivalent (in the sense of [16]) many-place ones (associated according to [20]) / "as well as the logics themselves" discovered here are especially valuable within the context of General Algebraic Logic going back to [13, 16, 17, 19]. In this connection, the "implicative" analogue of Theorem 15 of [19] — Lemma 4.14 — being essentially due to that of Lemma 11 therein — Lemma 4.12 — looks especially remarkable.

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