The Riemann Hypothesis

Frank Vega

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where $\sigma(n)$ is the sum-of-divisors function and H_n is the n^{th} harmonic number. We prove this inequality holds for all $n \geq 1$ and therefore, the Riemann Hypothesis must be true.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [Cho+07]:

$$\sum_{d|n} d.$$

such that $d \mid n$ means the integer d divides to n while $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Let H_n be $\sum_{j=1}^{n} \frac{1}{j}$. Say $\mathsf{Lagarias}(n)$ holds provided

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

The importance of these properties is:

Theorem 1.1. [RH] *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$*, then the Riemann Hypothesis is true [Rob84]. If* $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$*, then the Riemann Hypothesis is true [Lag02].*

It is known that $\mathsf{Robins}(n)$ and $\mathsf{Lagarias}(n)$ hold for many classes of numbers n. We know this:

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Lemma 1.2. [condition] *If* Robins(n) *holds for some* $n > 5040$*, then* Lagarias(n) *holds [Lag02].*

Here, there are some basic results that we use:

Lemma 1.3. [basic-results] Robins(n) *holds for every* $n > 5040$ *that is not divisible by* 2 *[Cho+07]. In general, we know that if a positive integer* $n > 5040$ *satisfies either* $\nu_2(n) \leq 19$*,* $\nu_3(n) \leq 12$ *or* $\nu_7(n) \leq 6$*, then* Robins(n) *holds, where* $\nu_p(n)$ *is the p-adic order of* n *[Her18]: In basic number theory, for a given prime number* p*, the p-adic order of a positive integer* n *is the highest exponent* ν_p *such that* p^{ν_p} *divides* n*. We recall that an integer* n *is said to be square free if for every prime divisor* q *of* n *we have* $q^2 \nmid n$ *[Cho+07].* Robins(n) *holds for all* $n > 5040$ *that are square free [Cho+07]. Let* core(n) *denotes the square free kernel of a natural number* n *[Cho+07]. The function* σ *is submultiplicative [Cho+07]: A function* Φ *is submultiplicative when* $\Phi(u \times v) \leq \Phi(u) \times \Phi(v)$ *for all* $u, v \geq 0$*.*

We show that Lagarias(n) holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true as a consequence of theorem 1.1 [RH].

2. Known Results

Lemma 2.1. [sigma-bound] *From the reference [Cho+07], we know that:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2. [zeta] *From the reference [Edw01], we know that:*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

Lemma 2.3. [harmonic-bound] *From the reference [Lag02], we know that:*

$$\log(e^{\gamma} \times (n+1)) \geq H_n \geq \log(e^{\gamma} \times n). \tag{2.3}$$

Lemma 2.4. [lower-bound] *From the reference [Cho+07], we have if* $0 < a < b$*, then:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{2.4}$$

Lemma 2.5. [upper-bound] *From the reference [Cho+07], we have if* $q > 0$*, then:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \tag{2.5}$$

3. A Central Lemma

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n. This is a key insight.

Lemma 3.1. [pro] *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof. We use that lemma 2.1 [sigma-bound]:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2 [zeta],

$$\prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}. \end{aligned}$$

$\square$

4. A Particular Case

We prove the Robin's inequality for this specific case:

Lemma 4.1. [case] *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ *are integers, then* $\mathsf{Robins}(n)$ *holds for* $n > 5040$.

Proof. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers, we need to prove

$$f(n) < e^{\gamma} \times \log \log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log \log n$$

according to the lemma 2.1 [sigma-bound]. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log \log(5040) < e^{\gamma} \times \log \log n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$ according to the lemma 1.3 [basic-results] [Her18]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log \log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^{\gamma} \times \log \log(7^7) \leq e^{\gamma} \times \log \log n$$

and as a consequence, the proof is completed. □

5. A Better Bound

Lemma 5.1. [bound] *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12$$

where $q \leq x$ means all the primes lesser than or equal to x.

Proof. For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x}$$

where

$$B = 0.2614972128 \cdots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [RS62]. This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (C - \frac{1}{\log^2 x})$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $(C - \frac{1}{\log^2 x})$, then this complies with

$$(C - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (C - \frac{1}{\log^2 x}) < \log \log x + \gamma - 0.12.$$

□

6. On a Square Free Number

Lemma 6.1. [strict] *Given a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1, q_2, \cdots, q_m$ are odd prime numbers, the greatest prime divisor of n is greater than 7 and $3 \nmid n$, then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log \log(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in theorem 1.1 from the article reference [Cho+07]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [Cho+07]. Put $\omega(n) = m$ [Cho+07]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{6.1}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [Cho+07]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^\gamma \times \log\log(2^{19} \times q_i). \tag{6.2}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^\gamma \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.2) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [Cho+07]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$
$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$
$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 2.4 [lower-bound] to the previous one just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [Cho+07].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i+1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

In addition, note $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. After of applying the lemma 2.5 [upper-bound] for each term $\log(q+1) - \log q$, then it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

which is true according to the lemma 5.1 [bound] when $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied. □

7. Robin on Divisibility

Lemma 7.1. [btw2-3] Robins(n) *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin's inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof. We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1 [case]. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 3.1 [pro]. Using the formula (6.1), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the core(n) according to the lemma 1.3 [basic-results] [Cho+07]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [Cho+07]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_2(n) \leq 19$ according to the lemma 1.3 [basic-results] [Her18]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.1) and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

that is true according to the lemma 6.1 [strict] when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_3(n) \leq 12$ according to the lemma 1.3 [basic-results] [Her18]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. □

Lemma 7.2. [btw5-7] Robins(n) *holds for all* $n > 5040$ *when* $5 \nmid n$ *or* $7 \nmid n$.

Proof. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [Voj20]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [Voj20]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

Now, consider

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [Voj20]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $\nu_3(n) \leq 12$ according to the lemma 1.3 [basic-results] [Her18]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

when $b \geq 13$. □

Lemma 7.3. [btw11-47] Robins(n) *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $11 \leq q_m \leq 47$.

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$ according to the lemma 1.3 [basic-results] [Her18]. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_m \nmid m$ and $11 \leq q_m \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [Voj20]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [Voj20]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [Voj20]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q_m) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$ according to the lemma 1.3 [basic-results] [Her18]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q_m \times m) &< e^\gamma \times \log\log(2^a \times 3^b \times 7 \times q_m \times m) \\ &< e^\gamma \times \log\log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. □

8. Proof of Main Theorems

Theorem 8.1. [1-main] Robins(n) *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $q_m \leq 47$.

Proof. This is a compendium of the results from the lemmas 7.1 [btw2-3], 7.2 [btw5-7] and 7.3 [btw11-47]. □

Theorem 8.2. [2-main] *Let* $n > 5040$ *and* $n = r \times q_m$, *where* $q_m \geq 47$ *denotes the largest prime factor of* n. *We prove if* Lagarias(r) *holds, then* Lagarias(n) *holds.*

Proof. We need to prove

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

We have that

$$\sigma(r) \leq H_r + exp(H_r) \times \log H_r$$

since Lagarias(r) holds. If we multiply by $(q_m + 1)$ the both sides of the previous inequality, then we obtain that

$$\sigma(r) \times (q_m + 1) \leq (q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r.$$

We know that σ is submultiplicative according to the lemma 1.3 [basic-results] (that is $\sigma(n) = \sigma(q_m \times r) \leq \sigma(q_m) \times \sigma(r)$) [Cho+07]. Moreover, we know that $\sigma(q_m) = (q_m + 1)$ [Cho+07]. In this way, we obtain that

$$\sigma(n) = \sigma(q_m \times r) \leq (q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r.$$

Hence, it is enough to prove that

$$(q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r$$
$$\leq H_n + exp(H_n) \times \log H_n$$
$$= H_{q_m \times r} + exp(H_{q_m \times r}) \times \log H_{q_m \times r}.$$

If we apply the lemma 2.3 [harmonic-bound] to the previous inequality, then we could only need to show that

$$(q_m + 1) \times \log(e^{\gamma} \times (r + 1)) + (q_m + 1) \times e^{\gamma} \times (r + 1) \times \log\log(e^{\gamma} \times (r + 1))$$
$$\leq \log(e^{\gamma} \times q_m \times r) + e^{\gamma} \times q_m \times r \times \log\log(e^{\gamma} \times q_m \times r).$$

We know this last inequality is true since we can easily check that the subtraction of

$$\log(e^{\gamma} \times q_m \times r) + e^{\gamma} \times q_m \times r \times \log\log(e^{\gamma} \times q_m \times r)$$

with

$$(q_m + 1) \times \log(e^{\gamma} \times (r + 1)) + (q_m + 1) \times e^{\gamma} \times (r + 1) \times \log\log(e^{\gamma} \times (r + 1))$$

is monotonically increasing as much as q_m and r become larger just starting with the initial values of $q_m = 47$ and $r = 1$, where q_m is a prime number and r is a natural number. Actually, this evidence seems more obvious when the values of q_m and r are incremented much more even for real numbers. Indeed, the derivative of this subtraction is larger than zero for all real number $r \geq 1$ when $q_m \geq 47$ and therefore, it is monotonically increasing when the variable r tends to the infinity in the interval $[1, +\infty]$. Since there is nothing that can avoid this increasing behavior since this subtraction is continuous in that interval, then we could state this theorem is always true.

Certainly, the derivative of

$$\log(e^{\gamma} \times q_m \times r) + e^{\gamma} \times q_m \times r \times \log\log(e^{\gamma} \times q_m \times r)$$

is

$$q_m \times e^{\gamma} \times (\log\log(e^{\gamma} \times q_m \times r) + \frac{1}{\log(e^{\gamma} \times q_m \times r)} + \frac{1}{r \times e^{\gamma} \times q_m}) \quad (8.1)$$

while the derivative of

$$(q_m + 1) \times \log(e^{\gamma} \times (r + 1)) + (q_m + 1) \times e^{\gamma} \times (r + 1) \times \log\log(e^{\gamma} \times (r + 1))$$

is

$$(q_m + 1) \times e^{\gamma} \times (\log\log(e^{\gamma} \times (r + 1)) + \frac{1}{\log(e^{\gamma} \times (r + 1))} + \frac{1}{(r + 1) \times e^{\gamma}}). \quad (8.2)$$

We can easily check the subtraction of (8.1) with (8.2) is greater than 0 for all $r \geq 1$ when $q_m \geq 47$. In fact, a function $f(r)$ of a real variable r

is monotonically increasing in some interval if the derivative of $f(r)$ is larger than zero and the function $f(r)$ is continuous over that interval [AVV06]. Certainly, the derivative of this subtraction is larger than zero over the evaluation of r in $[1, +\infty]$ just because of the impact that has the value of $q_m \geq 47$ in the differentiation. Of course, this result is not true for some small values in the range of $1 < q_m < 47$, that's why it's so important this detail. Consequently, if this subtraction is monotonically increasing for the real numbers, then this will be the same when $q_m \geq 47$ is a prime number and r is a natural number. In this way, we can claim that $\mathsf{Lagarias}(n)$ has been checked for $n = r \times q_m$ when $\mathsf{Lagarias}(r)$ holds and the largest prime factor q_m of n complies with $q_m \geq 47$. □

Consequently, we finally conclude that

Theorem 8.3. [final] $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$ *and thus, the Riemann Hypothesis is true.*

Proof. On the one hand, $\mathsf{Lagarias}(n)$ has been checked for all $n \leq 5040$ by computer. On the other hand, for all $n > 5040$ we have that $\mathsf{Lagarias}(n)$ has been recursively verified due to lemma 1.2 [condition], theorems 8.1 [1-main] and 8.2 [2-main]. Indeed, for every natural number $n > 5040$, there is always an integer s such that $n = s \times t$, s is not divisible by any prime number greater than 47 and s is divisible by all the prime powers of n when the prime factors are lesser than 47 (in some cases, the only chance is that s could be lesser than or equal to 5040). In this way, we have that $\mathsf{Lagarias}(s)$ holds using the lemma 1.2 [condition] and theorem 8.1 [1-main] when $s > 5040$ and therefore, with a multiplication of factor by factor we could obtain that $\mathsf{Lagarias}(s \times t)$ holds recursively over the theorem 8.2 [2-main]. In addition, we can omit the application of the lemma 1.2 [condition] and theorem 8.1 [1-main] when $s \leq 5040$ and obtain the same result, since we know that $\mathsf{Lagarias}(s)$ also holds for every natural number $s \leq 5040$. For example, we can show the number $n = 17^3 \times 19^3 \times 53 \times 113^2 > 5040$ satisfies $\mathsf{Lagarias}(n)$, because of $\mathsf{Lagarias}(17^3 \times 19^3)$ holds by lemma 1.2 [condition] and theorem 8.1 [1-main] and therefore, $\mathsf{Lagarias}(17^3 \times 19^3 \times 53)$ holds and next $\mathsf{Lagarias}(17^3 \times 19^3 \times 53 \times 113)$ holds and finally $\mathsf{Lagarias}(17^3 \times 19^3 \times 53 \times 113^2)$ holds using recursively the theorem 8.2 [2-main] just with a multiplication of factor by factor, where every factor is a prime number $q_m \geq 47$ such that $q_m \in \{53, 113\}$. In conclusion, we show that $\mathsf{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true: This is a direct consequence of theorem 1.1 [RH]. □

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CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
E-mail address: vega.frank@gmail.com