Entropic Hopf algebras

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1 Introduction

The concept of a Hopf algebra originated in topology. Classically, Hopf algebras are defined on the basis of unital modules over commutative, unital rings. The intention of the present work is to study Hopf algebra formalism (§1.2) from a universal-algebraic point of view, within the context of entropic varieties. In an entropic variety, the operations of each algebra are homomorphisms, and tensor products are well-behaved (§1.1, compare [1]). Entropic varieties include the classes of (pointed) sets, barycentric algebras [6, 7, 8], semilattices, and commutative monoids, as well as the classical example of modules over commutative rings. Hopf algebras over barycentric algebras and semilattices incorporate both coalgebraic and ordered structures. They offer an algebraic framework for the study of symmetry in these contexts.

In any entropic variety, the concept of a setlike (or grouplike) element may be defined, and group (Hopf) algebras constructed (§2). In particular, group Hopf algebras within the variety of barycentric algebras consist precisely of finitely supported probability distributions on groups.

For primitive elements and group quantum doubles, the natural universal-algebraic classes are entropic Jónsson-Tarski varieties (such as semilattices or commutative monoids) (§3). There, the free monoid or tensor algebra on any algebra is a bi-algebra, and the set of primitive elements of a Hopf algebra forms an abelian group.

In general, we use the algebraic notations and conventions of [10]. For classical Hopf algebras (or "quantum groups"), one may refer to [5, 11].

1.1 Tensor products

Let $\mathbf{V}$ be a variety of entropic algebras, considered as a category with homomorphisms as morphisms. For algebras Z, Y, X in $\mathbf{V}$, the set $\mathbf{V}(Y, X)$ of homomorphisms from Y to X is a subalgebra of the power X^Y with pointwise operations inherited from X. The *tensor product* $Z \otimes Y$ of Z and Y is defined by the adjoint relationship $\mathbf{V}(Z \otimes Y, X) \cong \mathbf{V}\big(Z, \mathbf{V}(Y, X)\big)$ [1, 4]. Currying yields a natural map $\otimes\colon Z \times Y \to Z \otimes Y; (z, y) \mapsto z \otimes y$. The commutativity of the tensor product, in the form of an isomorphism $\tau\colon Z \otimes Y \to Y \otimes Z$ for algebras Y, Z in $\mathbf{V}$, is obtained as a consequence [10, III(3.6.6)]. The general associativity of the tensor product in an entropic variety is discussed in [1, §3], and will be used implicitly throughout.

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Proposition 1.1. *Let* $\mathbf{1}$ *be the free algebra in* $\mathbf{V}$ *on one generator* $\{x\}$.

(a) *There are natural isomorphisms* $\mathbf{1} \otimes A \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} A \otimes \mathbf{1}$ *for each algebra* A *in* $\mathbf{V}$.

(b) *The free algebra* $\mathbf{1}$ *carries a commutative monoid multiplication*

$$\nabla \colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}; x \otimes x \mapsto x$$

and a comultiplication $\Delta \colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}$ *as a pair of mutually inverse* $\mathbf{V}$*-homomorphisms.*

1.2 Hopf algebras

Definition 1.2. Let $\mathbf{V}$ be an entropic variety.

(a) A *monoid* in $\mathbf{V}$ is a $\mathbf{V}$-algebra A equipped with a $\mathbf{V}$-homomorphism $\nabla \colon A \otimes A \to A$ known as *multiplication*, and a $\mathbf{V}$-homomorphism $\eta \colon \mathbf{1} \to A$ known as the *unit*, such that $(\nabla \otimes 1_A)\nabla = (1_A \otimes \nabla)\nabla$, $(\eta \otimes 1_A)\nabla = \lambda_A$, and $(1_A \otimes \eta)\nabla = \rho_A$. A monoid is *commutative* if $\tau\nabla = \nabla$.

(b) A *comonoid* in $\mathbf{V}$ is a $\mathbf{V}$-algebra A equipped with $\mathbf{V}$-homomorphisms $\Delta \colon A \to A \otimes A$ known as *comultiplication*, and $\varepsilon \colon A \to \mathbf{1}$ known as the *counit*, such that $\Delta(\Delta \otimes 1_A) = \Delta(1_A \otimes \Delta)$, $\Delta(\varepsilon \otimes 1_A) = \lambda_A^{-1}$, and $\Delta(1_A \otimes \varepsilon) = \rho_A^{-1}$. A comonoid is *co-commutative* if $\Delta\tau = \Delta$.

(c) A *bi-algebra* in $\mathbf{V}$ is a monoid and comonoid A in $\mathbf{V}$ such that the comonoid operations are monoid homomorphisms.

(d) A *Hopf algebra* in $\mathbf{V}$ is a bi-algebra A in $\mathbf{V}$ with a $\mathbf{V}$-homomorphism $S \colon A \to A$ known as the *antipode*, such that $\Delta(S \otimes 1_A)\nabla = \varepsilon\eta = \Delta(1_A \otimes S)\nabla$.

Remark 1.3. Comonoids are coalgebras in the categorical sense, and one may study their coalgebra congruences [2, Lemma 4.12]. However, while the quotient of a comonoid by a congruence is a comonoid, there are comonoid homomorphisms whose kernels are not coalgebra congruences. The question of a First Isomorphism Theorem for general comonoids or Hopf algebras remains open.

2 Setlike elements and group Hopf algebras

Let A be a comonoid in an entropic variety $\mathbf{V}$. Let $\mathbf{1}$ be the free $\mathbf{V}$-algebra on $\{x\}$. Then an element a of A is said to be *setlike* [11, p.40] (or "grouplike" [5, p.8]) if $a\Delta = a \otimes a$ and $a\varepsilon = x$.

Proposition 2.1. *Let* A_1 *be the set of setlike elements of a bi-algebra* A *in an entropic variety* $\mathbf{V}$, *with unit* $\eta \colon \mathbf{1} \to A; x \mapsto 1$.

(a) *The structure* $(A_1, \cdot, 1)$ *forms a monoid under the multiplication given by* $a \cdot b = (a \otimes b)\nabla$.

(b) *Suppose that* A *is a Hopf algebra in* $\mathbf{V}$. *Then* $(A_1, \cdot, 1)$ *is a group, with inversion given by the antipode* S.

Theorem 2.2. *Let* $\mathbf{V}$ *be an entropic variety, and let* G *be a group. Let* GV *be the free* $\mathbf{V}$*-algebra over the set* G. *Then* GV *carries the structure of a Hopf algebra in* $\mathbf{V}$, *in which the images of the elements of* G *in* GV *are setlike. The multiplication and antipode are the respective extensions of the group multiplication and inversion.*

Example 2.3. Of particular interest is the case where $\mathbf{B}$ is the variety of barycentric algebras. The free barycentric algebra GB on the set G is the set

$$\left\{ \sum_{i=1}^{n} p_i g_i \;\middle|\; 0 < n \in \mathbb{Z}, 0 \leq p_1, \dots, p_n \in \mathbb{R}, \sum_{i=1}^{n} p_i = 1 \right\}$$

of all convex linear combinations of elements of the set G. Note that GB is a convex subset of the real group algebra $\mathbb{R}G$. The group algebra structure on GB that is given by Theorem 2.2 is inherited from the Hopf algebra $\mathbb{R}G$. If G is finite, GB consists precisely of all the probability distributions on G. More generally, it consists of all the finitely supported probability distributions on an arbitrary group G.

3 Jónsson-Tarski varieties

A variety of universal algebras is a *Jónsson-Tarski variety* if there is a derived binary operation $+$ and nullary operation selecting a subalgebra $\{0\}$ such that the identities $0 + a = a = a + 0$ hold [3, 9]. Here, the Jónsson-Tarski varieties under consideration are entropic, e.g. modules over commutative rings, commutative monoids, or semilattices. If A is an algebra in an entropic Jónsson-Tarski variety, then $(A, +, 0)$ is a commutative monoid.

3.1 Primitive elements

Let A be a comonoid in an entropic Jónsson-Tarski variety $\mathbf{V}$. Suppose that $\mathbf{1}$ is the free $\mathbf{V}$-algebra on $\{x\}$, and that $\eta \colon \mathbf{1} \to A; x \mapsto 1$ is a $\mathbf{V}$-morphism. Then an element a of A is said to be *primitive* if $a\Delta = a \otimes 1 + 1 \otimes a$ and $a\varepsilon = 0$.

Proposition 3.1. *Let A_0 be the set of primitive elements of A. Then A_0 is a submonoid of $(A, +, 0)$.*

Theorem 3.2. *If A is a Hopf algebra in $\mathbf{V}$, then $(A_0, +, 0)$ is an abelian group, with inversion given by the antipode S.*

3.2 Tensor algebras

Let A be an algebra in an entropic variety $\mathbf{V}$. Then the *tensor algebra AT* over A is the free monoid in $\mathbf{V}$ over A.

Theorem 3.3. *Suppose that A is an algebra in an entropic Jónsson-Tarski variety $\mathbf{V}$.*

(a) *The tensor algebra AT over A carries a uniquely defined bi-algebra structure in which each element of A is primitive.*

(b) *If the commutative monoid $(A, +, 0)$ is an abelian group, then the bi-algebra AT carrries the structure of a Hopf algebra in $\mathbf{V}$.*

(c) *If AT is a Hopf algebra in $\mathbf{V}$, then the commutative monoid $(A, +, 0)$ is cancellative.*

Problem 3.4. In the context of Theorem 3.3(c), is $(A, +, 0)$ necessarily an abelian group?

3.3 Quantum doubles

For a finite group G, the Hopf algebra D that is constructed in the following theorem is known as the *quantum double* of G (compare [12]). In the variety $\mathbf{V}$, the free $\mathbf{V}$-algebra on a set X is written as XV.

Theorem 3.5. *Let* $\mathbf{V}$ *be an entropic Jónsson-Tarski variety. Let* G *be a finite group with identity element* e*. Set* $D = GV \otimes GV$*, and write* $h|g$ *for* $h \otimes g$ *with elements* g *and* h *of* G*. Define* $(h|f \otimes k|g)\nabla = hk|g$ *if* $fk = kg$*, and* 0 *otherwise. Define a unit* $\eta\colon x \mapsto \sum_{g\in G} e|g$*. Define an antipode* $S\colon h|g \mapsto h^{-1}|hg^{-1}h^{-1}$ *and comultiplication* $\Delta\colon h|g \mapsto \sum_{g^L g^R = g} h|g^L \otimes h|g^R$*. Define* $(h|g)\varepsilon = x$ *if* $g = e$*, and* 0 *otherwise. Then* $(D, \nabla, \eta, \Delta, \varepsilon, S)$ *is a Hopf algebra in* $\mathbf{V}$*.*

Theorem 3.5 furnishes Hopf algebras D that are neither commutative nor cocommutative, whenever the group G is non-abelian.

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